〈論　説〉

A Simple Survey of Dynamic Programming and Applications

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Abstract

In this paper, I introduce the development of dynamic programming. To solve dynamic programming numerically, function approximation and numerical integration are required. The purpose of this paper is to provide a brief survey of them. Furthermore, as a training application of dynamic programming, we introduce a labor market search model and develop a series of dynamic programming operations.

1. Introduction

In economic analysis, more advanced analysis is required due to developments in computers. This is also influenced by the emergence of free software such as Python and Julia that can implement advanced calculations.

In this paper, we review the theoretical development of dynamic programming (DP), numerical analysis, and simulation methods. Since Bellman (1957), DP has become a powerful tool for solving dynamic models. As just said, there are many cases in dynamic economic analysis that show numerical simulations. There are many different mathematical techniques to make this possible, but advances in analysis software are helping to drive down the cost of doing so. In this paper I present a representative model approximation and numerical integration.

A useful exercise for these sequences of deployment is the job search model. In particular, it makes a lot of sense as an implementation training for DP on discrete choice problems. In this regard, Thomas Sargent and John Stachurski provide one programming package "QuantEcon" , which makes it even easier to work with.

2. DP and Simulations

DP, known as backward induction, is widely used in economics and has been an important tool forsolving optimization problems. It is a recursive method for solving sequential decision problems and applies to both discrete and continuous time models. According to Bellman (1957), the DP problem has two important variables (vectors): the state variables and the activity variables (decision variables or control variables). Optimal decision making is a function that depends on time as well as state variables called policy functions. Bellman (1957) states that an optimal policy constitutes an optimal policy with respect to the state in which the remaining decisions arise from the initial decision, whatever the initial state and initial decision. The equivalence between the original sequential decision problem and its corresponding DP problem is called the optimality principle.

2.1 Definition of decision-making process and Bellman's equation

Now consider the discrete time problem. In this case, the Markov decision making process (MDP) consists of a time index $t \in \{0, 1, \dots, T\}$ $(T \leq \infty)$, a state space S, an activity space A, a constraint set $\{A(S) \subseteq A\}$, a transition probability $p(ds'|s, a)$, a discount factor $\beta \in (0, 1)$, and an additively separable reward function[1]

$$R_T(\mathbf{s}, \mathbf{d}) = \sum_{t=0}^{T} \beta^t r(s_t, a_t). \tag{1}$$

The optimization problem is to choose the optimal rule $\pi = (\pi_0, \ldots, \pi_T)$ for decision making to solve the following problem:

$$\max_{\pi} \mathbb{E}_{\pi}\{_T(\mathbf{s}, \mathbf{d})\} \equiv \int_{s_0} \cdots \int_{s_T} \left[\sum_{t=0}^{T} \beta^t (rs_t, \pi_t(s_t)) \right] \prod_{t=1}^{T} p(ds_t | s_{t-1}, \pi_{t-1}(s_{t-1})) p_0(ds_0) \tag{2}$$

where, p_0 is the probability distribution from which the initial state s_0 is drawn. The problem is that solving eq. (2) requires $T + 1$ multivariate integrations, which is impossible to implement. Then, using DP, I can greatly simplify this intractable optimization problem.

First, I consider the case of a finite time horizon ($T < \infty$). DP is a simple backward induction. At the final time T,

$$\pi_T(s_T) = \arg\max_{a_T \in A(s_T)} [r(s_T, a_T)], \quad and$$
$$v_T(s_T) = \max_{a_T \in A(s_T)} [r(s_T, a_T)]$$

are defined. By induction,

$$\pi_t(s_t) = \arg\max_{a_t \in A(s_t)} \left[r(s_t, a_t) + \beta \int v_{t+1}(s_{t+1}) p(ds_{t+1} | s_t, a_t) \right]$$
$$v_t(s_t) = \max_{a_t \in A(s_t)} \left[r(s_t, a_t) + \beta \int v_{t+1}(s_{t+1}) p(ds_{t+1} | s_t, a_t) \right]$$

are defined for $t = 0, 1, \ldots, T - 1$. I can confirm that the value function $v_0(s_0)$ at time $t = 0$ represents the expected discounted value of the maximized reward at all future time points. I have

$$v_0(s) = \max_{\pi} \mathbb{E}_{\pi}\{R_T(\tilde{\mathbf{s}}, \tilde{\mathbf{d}}) | s_0 = s\} \tag{3}$$

because DP recursively generates the optimal decision rule $\pi = (\pi_0, \ldots, \pi_T)$.

Next, I consider the case of an infinite time horizon with no end point ($T = \infty$). However, if the reward function r is bounded and the discount factor is $\beta \in (0, 1)$, $R_\infty(\mathbf{s}, \mathbf{d})$ can be approximated to the reward function $R_T(\mathbf{s}, \mathbf{d})$ of the finite time horizon problem with T large enough. This is the basic concept for numerically solving the infinite time horizon DP.

Stationarity is required for MDP for infinite period. This means that the transition probabilities and reward functions are the same for all t. This defines v and π of infinite duration as

$$\pi(s) = \arg\max_{a \in A(s)} \left[r(s,a) + \beta \int v(s')p(ds'|s,a) \right] \tag{4}$$

$$v(s) = \max_{a \in A(s)} \left[r(s,a) + \beta \int v'(s)p(ds'|s,a) \right] \tag{5}$$

by omitting the subscript t. The functional eq. (4) is known as the Bellman equation and the value function v is the fixed point of this functional equation.

To check that the solution v of the Bellman equation exists and is unique, suppose (1) the S and A are complete metric spaces[2], and (2) $r(s, a)$ is continuous in (s, a). (3) $s \to A(s)$ is of continuous correspondence. Let $\mathscr{B}(S)$ be the Banach space of all measurable and bounded functions $f : S \to \mathbb{R}$ under the sup norm $||f|| = \mathbf{sup}_{s \in S}|f(s)|$ [3]. Define the Bellman operator $\Gamma : \mathscr{B}(S) \to \mathscr{B}(S)$ by

$$\Gamma(w)(s) = \max_{a \in A(s)} \left[r(s,a) + \beta \int w(s')p(ds'|s,a) \right]. \tag{6}$$

Therefore, the Bellman equation can be rewritten as

$$v = \Gamma(v). \tag{7}$$

In other words, v is a fixed point of the mapping Γ. Denardo (1967) shows that the Bellman operator has a particularly good property of contraction mapping. This implies that for w and v in a compact state space $\mathscr{B}$,

$$||\Gamma(v) - \Gamma(w)|| \leq \beta ||v - w||. \tag{8}$$

The contraction mapping establishes the existence and uniqueness of the solution v to the Bellman equation.

The uniqueness of fixed points is a direct consequence of the contraction property (8). If w and v are fixed points for Γ, eq. (8) implies that

$$||v - w|| = ||\Gamma(v) - \Gamma(w)|| \leq \beta ||v - w||.$$

Since $\beta \in (0, 1)$, the only possible solution to this equation is $||v - w|| = 0$. The existence of fixed points is due to the completeness of Banach spaces. Starting from some initial element (say 0) in Banach space, the contraction (8) implies that the following approximation sequence forms the Cauchy sequence in Banach space.

$$\left\{0, \Gamma(0), \Gamma^2(0), \Gamma^3(0), \ldots, \Gamma^n(0), \ldots\right\}$$

Since the Banach space $\mathscr{B}$ is complete, the Cauchy sequence converges to the point $v \in \mathscr{B}$. Thus existence is established by showing that v is a fixed point of Γ.

Using the contractility of Γ, it is known that for an infinite horizon problem, the following holds for some bounded and continuous function v_0:

$$v = \lim_{n \to \infty} \Gamma^n(v_0) = \lim_{n \to \infty} \Gamma\left(\Gamma^{n-1}(v_0)\right) = \Gamma(v_0)$$

where v is a fixed point of Γ and Γ^n denotes n iterations of the Bellman operator Γ.

I then establish that the stationary decision-making rule defined by v in eq (4) is optimal. The Blackwell theorem is a necessary theorem for this purpose. This theorem constructs that the stationary Markovian infinite policy given by π in eqs. (4) and (5) produces the optimal decision making rule for eq. (2), the infinite MDP problem. This theorem is a sufficient condition for further discussion of fixed points. Here, the Blackwell theorem requires that Γ be (1) monotonic[4] and (2) discounted[5].

Finally, I see approximate fixed point error bounds for approximating the Bellman operator. In problems S contains infinite states, the value function v is usually approximated simply by being an element of an infinite-dimensional Banach space $\mathscr{B}(S)$. In general, to approximate the fixed point v = $\Gamma(v)$ of $\mathscr{B}(S)$, the fixed point $v_N = \Gamma_N(v_N)$ of the approximate Bellman operator $\Gamma_N : B_N \to B_N$ is computed. where B_N is a finite-dimensional subset of $\mathscr{B}(S)$. The following lemma provides the error bounds necessary to prove the convergence of such approximations.

Consider $\{\Gamma_N\}$ is a contraction map on the N-indexed Banach space B that is point convergent. For $\forall w \in B$, we can obtain:

$$\lim_{n \to \infty} \Gamma_N(w) = \Gamma(w) \tag{9}$$

I can see $\lim_{N \to \infty} ||v_N - v|| = 0$ since the approximate fixed point $v_N = \Gamma_N(v_N)$ satisfies the error bound[6].

$$||v_N - v|| \leq \frac{||\Gamma_N(v) - \Gamma(v)||}{1 - \beta} \tag{10}$$

The value function iteration continues up to $||v_N - v|| < \varepsilon$ for $\varepsilon > 0$.

2.2 Approximation method

What is important in measuring the sequence problem is approximating the function. Representative methods include Taylor expansion, perturbation method, and interpolation method. Here we focus on the interpolation method.

Now, consider a continuous function $v : [a, b] \to \mathbb{R}$. The purpose of interpolation is to approximate v by $\hat{v}(x) = \Sigma c_j \phi_j(x)$, where $j = 0, \ldots, n$ is called the degree of interpolation, $\phi_j(x)$ is the basis function, and c_j is the basis coefficient. The first step in implementing the interpolation method is to select an appropriate basic function. Methods for doing this include the spectral method and the finite element method. The spectral method uses basis functions $\phi_j(x)$ that are globally non-zero, and the finite element method uses basis functions that are non-zero only in subregions of the approximation region $[a, b]$. Typical spectral methods include ordinary polynomial approximation and Chebyshev polynomial approximation. On the other hand, typical finite element methods include piecewise linear interpolation, cubic-splines, and B-splines.

The second step is to determine the basis coefficient c_j. A typical way to solve for $j = 1, \ldots, n$ coefficients is to find a polynomial such that $\hat{v}(x_i) = v_i$ for i given a data pair (x_i, v_i) (where $i = 1, \ldots, m$). In the simplest univariate case, the function $\hat{v}(x)$ is a polynomial of degree $m - 1$. That is, I find the polynomial coefficient c_i such that $\hat{v}(x) = \Sigma c_i x_i$ $(i = 0, \ldots, m - 1)$. Now, defining $\mathbf{c} = (c_0, c_1, \ldots, c_{m-1})'$, $\mathbf{v} = (v_1, \ldots, v_m)'$, and

$$\mathbf{A} = \begin{bmatrix} 1 & x_1 & \cdots & x_1^{m-1} \\ 1 & x_2 & \cdots & x_2^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & \cdots & x^{m-1} \end{bmatrix}, \tag{11}$$

these coefficients are theoretically calculated by $\mathbf{Ac} = \mathbf{v}$. Also, x^j is a general family of basis functions called monomial basis functions.

The problem here is how to find the n-th order polynomial approximation when $n < m$, and this problem can arise in dealing with multivariate problems. In this case, I can use the least-squares method solving the following problem.

$$\min_{\{c_i\}} \sum_{i=1}^{m} \left[v(x_i) - \sum_{j=0}^{n} c_j \phi_j(x_i) \right]^2 \tag{12}$$

Then, the solution to this minimization problem is $\mathbf{c} = (\boldsymbol{\Phi}'\boldsymbol{\Phi})^{-1}\boldsymbol{\Phi}'\mathbf{v}$, which is equivalent to interpolation if $n = m - 1$ and $\boldsymbol{\Phi}$ are nonsingular, where

$$\boldsymbol{\Phi} = \begin{bmatrix} 1 & \phi_1(x_1) & \cdots & \phi_n(x_1) \\ 1 & \phi_2(x_2) & \cdots & \phi_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \Phi_1(x_m) & \cdots & \phi_n(x_m) \end{bmatrix}$$

(1) *Orthogonal polynomial*

In the polynomial approximation above, I have to solve equation (12). However, using matrix (11) can cause problems because the monomial basis functions are not orthogonal to each other [7].

Therefore, the basis functions and approximate nodes are selected so that the matrix $\boldsymbol{\Phi}$ is orthogonal. By making such a choice, I can solve equation (12) because $\boldsymbol{\Phi}'\boldsymbol{\Phi}$ is a diagonal matrix. This is the concept of orthogonal polynomials.

First, define

$$< f, g >= \int_a^b f(x)g(x)w(x)dx \tag{13}$$

as the weighted inner product. The weighting function $w(x)$ on the interval $[a, b]$

always has a positive finite integer on the interval. Now, if $< f, g >= 0$, the functions f and g are orthogonal to the weighting function $w(x)$. Additionaly, if $< \phi_i, \phi_j >= 0$ for some $i \neq j$, then the family of polynomials $\{\phi_j(x)\}$ are mutually orthogonal polynomials.

Well-known orthogonal polynomials include Legendre polynomials, Chebyshev polynomials, Laguerre polynomials, and Hermite polynomials. Legendre polynomials has weighted functions $w(x) = 1$ on [-1, 1], Chebyshev polynomials $w(x) = (1 - x^2)^{-1/2}$ on [-1, 1], Laguerre polynomials $w(x) = e^{-x}$ on $[0, \infty)$, and the Hermite polynomial $w(x) = e^{-x^2}$ on $(-\infty, \infty)$. Here we briefly introduce Chebyshev polynomials[8].

The Chebyshev polynomial originated from verifying whether the n-fold angle of the cosine function exists[9]. In conclusion, $\cos n\theta$ is generally expressed as an n-th order polynomial of $\cos\theta$. Suppose $T_j(x) = \cos j\theta$. When $x \in [-1, 1]$, this $T_j(x)$ can be written as $T_j(x) = \cos(j \cos^{-1} x)$ [10]. If we recursively define the j-order polynomial like

$$T_0(x) = 1,\ T_1(x) = x,\ T_2(x) = 2x^2 - 1,\ T_3(x) = 4x^3 - 3x,\ \ldots,$$

we would understand that the following recurrence formula is satisfied.

$$T_{j+1}(x) = 2xT_j(x) - T_{j-1}(x), \quad j = 1, 2, \ldots$$

In Figure 1 I depict the Chebyshev polynomials T_1 through T_4 on [-1, 1].

Furthermore, $T_j(x)$ satisfies the following relation.

$$\int_{-1}^{1} T_j(x_k) T_i(x_k) \frac{dx}{\sqrt{1 - x^2}} = \int_0^m \cos i\theta \cos j\theta = \begin{cases} 0 & \text{if } i \neq j \\ m & \text{if } i = j = 0 \\ m/2 & \text{if } 0 < i = j \leq n \end{cases}$$

The Chebyshev series with $T_j(x)$ as the basis function is denoted by[11]:

$$\hat{v}_n(x) = \frac{c_0}{2} + \sum_{j=1}^{n} c_j T_j(x)$$

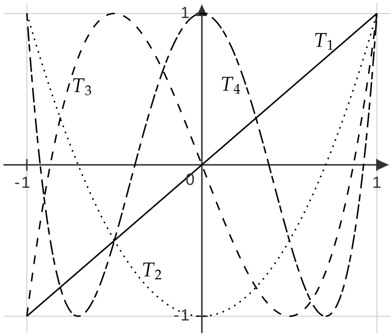


Fig.1. Chebyshev polynomial

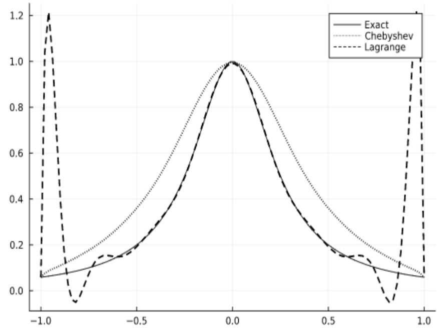


Fig.2. Interpolation compared

It is known that the Chebyshev polynomial $T_j(x)$ has the following mathematical properties. They are that $T_j(x) = 0$ on [-1, 1] has n solutions such that:

$$x_k = \cos\frac{(2k-1)\pi}{2n}, \quad k = 1, 2, \ldots, n$$

Additionally, there is a relationship of $T'_n(x) = uU_n(x)$ between Chebyshev polynomials of the first kind and the second kind, and the solutions (extreme values) with this being 0 are all of $x = \cos k\pi / n$ ($n = 0, 1, \ldots, n$). The y-coordinate given x is $T_n(\cdot) = \cos k\pi = (-1)^k$ and is -1 for odd k and +1 for even k. Combining this with the previous solution, I can decipt a graph between +1 and -1 as shown in Fig.1.

Finally, I have the Chebyshev interpolation as follow:

$$I^c_{n-1} = \sum^n c_i T_{i-1}(x) \tag{14}$$

Therefore, the (n - 1)-th order interpolation polynomial for the n-th order Chebyshev interpolation point ($x_1, \ldots, x_n$) is called a Chebyshev interpolation polynomial. It is known that the interpolation accuracy increases as the number of interpolation points increases. For comparison purposes, Fig.2 depicts the general Lagrangian interpolation and the Chebyshev interpolation with $n = 15$.

(2) *Spline interpolation*

Another useful interpolation method is spline interpolation[12]. Here is a brief introduction to the commonly used cubic splines. In Lagrangian interpolation, as the number of nodes increases, the function oscillates and the interpolation accuracy deteriorates (see Fig.2). Also, the spline method has the advantage of being easier to use than the Chebyshev interpolation. Spline interpolation is a method of dividing a region to be interpolated into data intervals $[x_j, x_{j+1}]$ and approximating them with a low-order polynomial using neighbore values. Of course, since I use an approximation function on the interval, there can be the problem of discontinuities in the derivative at the boundary if we do not properly approximate it.

In the cubic spline interpolation, the polynomial $S_j(x)$ is given by:

$$S_j(x) = a_j(x - x_j)^3 + b_j(x - x_j)^2 + c_j(x - x_j) + d_j, \quad j = 0, 1, \ldots, n - 1 \qquad (15)$$

It is important to obtain this coefficient in determining the spline interpolation function. The problem here is that for j, for example, there are 4 unknown coefficients, but only 2 equations. Therefore, it is necessary to set up sufficient constraint equations to calculate the unknowns according to the constraint conditions[13]. In the general case, there are $4n$ unknowns. Setting the constraints according to footnote.13, the smoothing constraints are $S_j(x_j) = y_j$, $S_j(x_{j+1}) = y_{j+1}$, $S_j'(x_j) = S_{j+1}'(x_j)$, and $S_j''(x_j) = S_{j+1}''(x_j)$, and the endpoint constraints are $S_0''(x_0) = 0$ and $S_{n-1}''(x_n) = 0$. This number is $2n + 2(n - 1) + 2 = 4n$, which is equal to the number of unknowns. By doing so, I can solve for the unknowns and perform the interpolation. Fig.3 depicts a cubic spline and a simple linear interpolation for $f(x) = \sin(x)$ over the interval [0, 8].

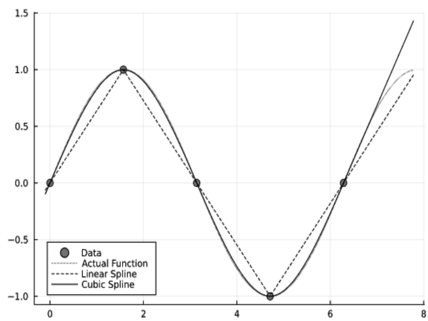


Fig.3. Linear and cubic spline interpolation

2.3 Numerical integration

In order to analytically solve DP under uncertainty and perform numerical simulations, it is necessary to derive the expectations of the model. It is related to how to handle the integral when calculating the expected value. Here, I briefly introduce how to compute the integral numerically[14].

One of the most well-known simulation methods is the Monte Carlo method. By the central limit theorem, the numerical error of the integral computed by the Monte Carlo method follows a normal distribution, which means that the numerical error generated by the Monte Carlo method is not bounded. Additionally, the optimization problem requires the evaluation of a large number of objective functions, and a wrong evaluation at some point makes the previous iterations meaningless. Therefore, it is necessary to use a numerical integration method with bounded numerical errors.

Numerical integration has several calculation methods depending on the probability distribution of random variables. Here, we briefly introduce the Gaussian quadrature method when the random variables follow the normal distribution and the uniform distribution. The basic concept of quadrature is to approximate an integral by a sum as follows:

$$\int_a^b f(x)dx \simeq \sum_{i=1}^{n} w_i f(x_i)$$

where x_i are the quadrature nodes and w_i is the quadrature weights. Finally, I consider the problem of finding x_i and w_i and approximating the definite integral by a weighted sum of the values $f(x_i)$. Here, I introduce the Gauss-Legendre formula and the Gauss-Hermitian formula.

The Gauss-Legendre formula gives good computational results when the random variables follow a uniform distribution. First, consider the weights w_i. When x_i $(i = 1, \dots, n)$ is given on $[a, b]$, the weights are written as:

$$w_i = \int_a^b \frac{\prod_{j\neq i}(x - x_j)}{\prod_{j\neq i}(x_i - x_j)} dx \tag{16}$$

Usually, we would expand the problem on [-1, 1][15]. Also, if x can be chosen appropriately, approximations up to order $2n$ - 1 can be performed accurately.

Now, let $x_1, \dots, x_n$ be the zero point of the n-th order Legendre polynomial $P_n(x)$, and calculate the weight from the Eq. (16)[16]. Then the following holds for any polynomial $f(x)$ up to degree $2n$ - 1.

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i) \tag{17}$$

This indicates that the integral value of the function $f(x)$ in [-1, 1] can be calculated by summing n values obtained by multiplying the function value at the quadrature nodes x_i by the weight w_i corresponding to the function value.

In fact, the weights and quadrature nodes for n have already been computed, e.g. $x_i = 0$ and $w_i = 2$ for $n = 1$, $x_i = \pm 0.577$ and $w_i = 1$ for $n = 2$, and, $x_i = 0, \pm 0.538, \pm 0.906$ and $w_i = 0.568, 0.478, 0.236$ for $n = 5$ [17].

Finally, we extend the state restricted to the interval [-1, 1] to the general interval $[a, b]$. This can be done by using the permutation integral. This leads to the following relationship.

$$\int_{a}^{b} f(x)dx = \frac{b-a}{2}\int_{0}^{1} f\left(\frac{(b-a)(x_i+1)}{2} + a\right) = \frac{b-a}{2}\sum_{i=1}^{n} w_i f\left(\frac{(b-a)(x_i+1)}{2} + a\right)$$

Next, consider the Gauss-Hermitian formula. This is a useful numerical integration method when the random variable follows $\mathcal{N}(\mu, \sigma^2)$. Just as in the Gauss-Legendre formula described above, the nodes are based on the zeros of the Hermite polynomials in this formula, just as the nodes are based on the zeros of the Legendre polynomials. In this formula, it follows that if, for example, for a random variable Y with distribution $\mathcal{N}(\mu, \sigma^2)$, one computes $\mathbb{E}f(Y)$, it suffices to compute:

$$\begin{aligned}\mathbb{E}f(Y) &= \left(2\pi\sigma^2\right)^{-1/2}\int_{-\infty}^{\infty} f\left(\sqrt{2}\sigma x + \mu\right)e^{-x^2}\sqrt{2}\sigma dx \\ &= \pi^{-1/2}\sum_{i=1}^{n} w_i f\left(\sqrt{2}\sigma x_i + \mu\right)\end{aligned}$$

Note that the second equation contains a weight function e^{-x^2}. So you can also understand that the weight function of the Gauss-Legendre formula is 1. See Stroud and Secrest (1966) for nodes and weights depending on the size of n.

3. Basics of Job Search Theory

Economic analysis of the labor market continues to develop as an important field. Among them, McCall (1970), which develops the most primitive model, considers three factors that affect the decision-making problem of the unemployed (job seekers): wage movements, discount factors, and unemployment benefits. In fact, extended models of McCall (1970) provide many exercises for dealing with DP. Therefore, such models are excellent tools for understanding dynamic programming and learning how to perform simulation analysis.

The basic job search model assumes that people are faced with discrete choices of accepting or rejecting job contract proposals. According to McCall (1970), in each period the unemployed draws one wage offer w from a cumulative distribution function $F(w) = \Pr\{w \leq \mathrm{W}\}$ such that $F(0) = 0$ and $F(B) = 1$ for a finite B. In other words, wages are non-negative random variables. Here, we briefly review the relationship between non-negative random variables and uncertainty, and introduce the underlying model of McCall (1970).

3.1 Uncertainty

Rothschild and Stiglitz (1970) expands the definition of mean-preserving spread (MPS) into a dynamic model. MPS defines the concept of risk expansion using second-order stochastic dominance (SOSD)[18].

Before defining MPS, let us first consider a non-negative random variable W. The expected value of a non-negative random variable W with cumulative distribution function $F(w)$ presented here is given by[19]:

$$\mathbb{E}(W) = \int_0^B \Pr[W > w]dw = \int_0^B [1 - F(w)]dw = B - \int_0^B F(w)dw \tag{18}$$

where $F(0) = 0$ and $F(B) = 1$ are used.

Next, we consider first-order stochastic dominance (FOSD) and SOSD, and examine the concept of MPS defined by Rothschild and Stiglitz (1970)[20]. Now suppose a general utility function such as $u'(w) > 0$ and $u''(w) < 0$. Let z_1 and z_2 be random variables and their density functions denotes $f_1(w)$ and $f_2(w)$. Comparing the expected values under these conditions yields

$$\int_0^B u(w)f_1(w,z)dw - \int_0^B u(w)f_2(w,z)dw = \int_0^B u'(w)[F_2(w,z) - F_1(w,z)]dw.$$

When this is equal to zero, it is said to be a homogeneous mean condition. In this equation, F_1 is said to have FOSD with respect to F_2 when $F_2(w) > F_1(w)$ for all $w \in (0, B)$.

Now, it is known that it can be written as follows:

$$F^{(2)}(w) = \int_0^w F(w)dw$$

This is the integral of the cumulative distribution function, so its value at any point w is the area under the $F(w)$ curve for w going from 0 to w. Also, by the fundamental theorem of the calculus, $F(w) = F^{(2)\prime}(w)$ for all w. The function $F^{(2)}$ is called as super-cumulative distribution function. Expanding in the same way as FOSD, between two different distributions,

$$\int_0^B u(w)f_1(w,z)dw - \int_0^B u(w)f_2(w,z)dw = \int_0^B u''(w)\left[F_1^{(2)}(w,z) - F_2^{(2)}(w,z)\right]dw$$

holds. This is positive only when $F_1^{(2)} < F_2^{(2)}$ for a utility function of general $u'' < 0$. Therefore, I can say that the distribution F_1 has SOSD to F_2 when $F_1^{(2)}(w) < F_2^{(2)}(w)$ for all $w \in (0, B)$ and $F_1^{(2)}(B) = F_2^{(2)}(B)$.

As Nishimura and Ozaki (2014) points out, if increased uncertainty in the labor market means increased risk to the wage distribution facing the unemployed (according to MSP), it will increase the reservation wages of the unemployed.

3.2 McCall model overview

The simple McCall's model does not assume recall. Let y_t be your income in period t. If you are unemployed, you will receive unemployment benefits c (i.e., $y_t = c$), and if you choose to work, you will receive wage income w (i.e., $y_t = w$). Unemployed workers living indefinitely strategize to maximize $\mathbb{E}\Sigma^{\infty}\ \beta^t y_t$. In this model, workers face a trade-off between (1) waiting longer for an offer and (2) accepting t=0 an offer earlier. The cost of (1) is due to the existence of the discount factor, and the cost of (2) arises from the uncertainty of future offers. The value function $v(w)$ is the expected utility of a decision maker who decides whether to accept or reject an offer w, and satisfies the following Bellman equation.

$$v(w) = \max\left\{\frac{w}{1-\beta}, c + \beta\int_0^B v(w')dF(w')\right\} \tag{19}$$

where the first term of the maximum operator is the future income from accepting the current offer w[21]. The second term is obtained through the activity of receiving unemployment benefits c in the current period when the offer was rejected and extracting the new wage w' from the distribution F of the next period.

Let us graph Eq.(19) using a simple numerical example. Here, we assume wages that follow a beta-binomial distribution with 100 trials and shape parameters α and β of 100 and 50, respectively (see Fig. 4). Assuming wages that follow this distribution, unemployment benefits are set at c = 30, the discount factor at β = 0.99. The result of calculating Eq.(19) based on this parameter is shown in Fig. 5.

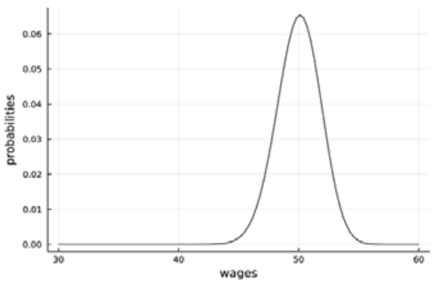


Fig.4: wage distribution

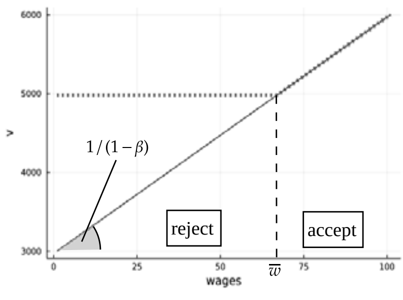


Fig.5: Relation between function (19) and $\overline{w}$

The solution to function (19) is defined as follows:

$$v(w) = \begin{cases} \dfrac{\overline{w}}{1-\beta} = c + \beta \displaystyle\int_0^B v(w')dF(w') & \text{if } w \leq \overline{w} \\ \dfrac{w}{1-\beta} & \text{if } w \geq \overline{w} \end{cases} \tag{20}$$

Using this equation, I can transform the functional Eq.(19) of the value function $v(w)$ into the ordering equation of the reservation wage $\overline{w}$. Using the first equation in Eq.(20), I can define the difference between the reservation wage and unemployment benefits as follows[22]:

$$\overline{w} - c = \frac{\beta}{1-\beta} \int_{\overline{w}}^B (w' - \overline{w})dF(w') \tag{21}$$

This formula is used to specify the reservation wage $\overline{w}$.

A brief consideration of Eq.(21) is given below. The left hand side of this equation is the cost of rejecting the wage offer $\overline{w}$ in order to continue the search. On the other hand, the right hand side is the expected discount benefit of rejecting $\overline{w}$ in order to continue searching. Eq.(21) therefore suggests that the reservation wage $\overline{w}$ be set so that the cost of another search equals the benefit.

Next, I show how the reservation wage is determined. Expanding Eq.(21), I can obtain[23]:

$$\overline{w} - c = \beta(\mathbb{E}w - c) + \beta g(\overline{w}) \tag{22}$$

Note that we define:

$$g(\overline{w}) = \int_0^{\overline{w}} F(w')dw'$$

This function has the properties $g(0) = 0$, $g(\overline{w}) \geq 0$, $1 \geq g'(\overline{w}) = F(\overline{w} > 0$, and $g''(\overline{w}) = F''(\overline{w}) > 0$ for $\overline{w} > 0$.

Here, let

$$g(w, z) = \int_0^w F(w', z)dw'$$

for some distribution function $F(w, z)$. Assuming the MPS above, we have the relation

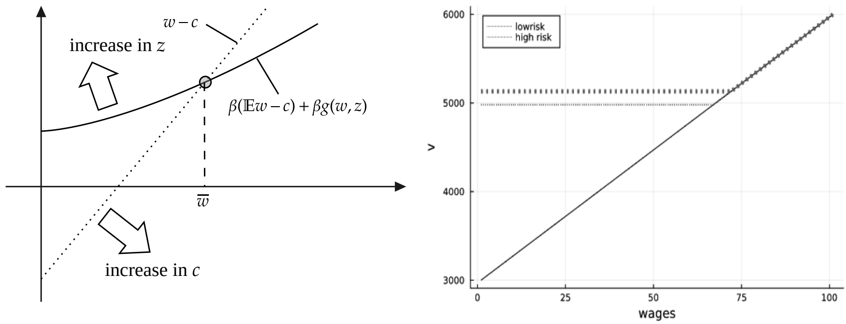


Fig. 6: Relationship between determination of $\overline{w}$ and uncertainty

$$\int_0^w [F_2(w', z) - F_1(w', z)]dw' \geq 0$$

between these distribution functions, so I have:

$$g_2(w, z) \geq g_1(w, z)$$

Finally, to derive the reservation wage $\overline{w}$, I need to solve

$$\overline{w} - c = \beta(\mathbb{E}w - c) + \beta g(\overline{w}, z) \tag{23}$$

This formula is depicted in Fig. 6.

Using the default parameters and wage distribution, let us calculate the fixed point equivalent of the reservation wage. Python and Julia contain packages that can compute this by iterative methods. Here, I ran 500 iterations, resulting in a reservation wage of 51.428.

Looking at Fig. 6, there are several implications involved. As simulated in Fig. 7, the reservation wage increases as the discount factor increases. Similarly, an increase in unemployment benefits will also increase the reservation wage. Moreover, as is clear from the right panel of Fig. 6, when risk z expands in a mean preserving, it also raises the reservation wage. This is because, in addition to increasing the probability of good wage offers, workers can reject bad offers. This result was also revealed by Nishimura and Ozaki (2014).

Finally, let us clarify other implications derived from the underlying model of McCall (1970) from simulations. They are the effects of discount factors and unemployment

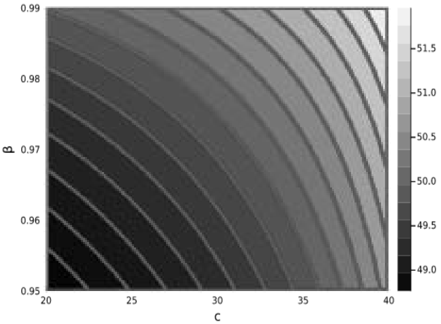


Fig. 7: Effect of c and β on $\overline{w}$

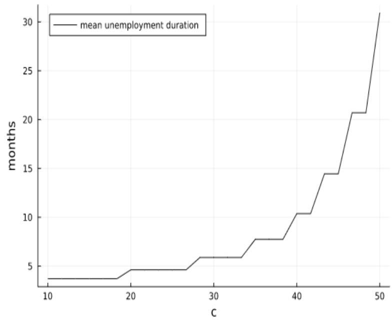


Fig. 8: c and the average duration of unemployment

benefits on reservation wages and the effect of unemployment benefits on average duration of unemployment. Their simulation results are depicted in Fig. 7 and Fig. 8. In particular, Fig. 8 is derived by simulation based on optimal stopping theory.
These represent the expected results. Looking at Figure 8, it is clear that increasing unemployment benefits lengthens the duration of unemployment. In addition, based on this model, it can be understood that when the amount of unemployment benefits increases, there is a high possibility that even a small increase in benefits will significantly extend the duration of unemployment.

4. Separations

So far, we have assumed that once a worker gets a job, he or she stays in that job permanently. However, this is unrealistic. Therefore, here we consider the M model that takes job separation into account.

Now suppose that workers face a separation rate $\alpha \in (0, 1)$ in each period. For simplicity, this separation rate is assumed to be independent of tenure t. A worker who is currently unemployed draws a wage offer w from a time-varying but known probability distribution F.

Let $\hat{v}(w)$ be the expected discounted present value of the income of the unemployed who receives an offer w. If an unemployed person accepts a wage offer, the expected discounted present value of his or her income is as follows:

$$\hat{v}(w) = w + \beta(1-\alpha)\hat{v}(w) + \beta\alpha\left[c + \beta\int \hat{v}(w')dF(w')\right]$$

Here, a worker who accepts a wage offer w receives w in the current period. Furthermore, since the probability of staying in the job in the next period is 1 - α, the discounted present value of income $\beta\hat{v}(w)$ is added with its probability. Workers, on the other hand, leave their jobs with probability α, receive unemployment benefits, and wait for new offers.

The value function is characterized as follows.

$$\hat{v}(w) = \begin{cases} c + \beta\int \hat{v}(w')dF(w') & \text{if } w \leq \overline{w} \\ \dfrac{w + \beta\alpha\left[c + \beta\int \hat{v}(w')dF(w')\right]}{1-\beta(1-\alpha)} & \text{if } w \geq \overline{w} \end{cases} \tag{24}$$

Reservation wages, taking into account separation, are obtained by solving:

$$\frac{\overline{w}}{1-\beta} = c + \beta\int \hat{v}(w')dF(w') \tag{25}$$

Note that this Eq is similar to the reservation wage in Eq.(20), but the value function is different. In general, since $\hat{v}(w)$ with the separation rate is smaller than $v(w)$ without it, the reservation wage taking into account the separation is lower.

Using the derivation process of Eq.(21), we can obtain[24]:

$$\overline{w} - c = \frac{\beta}{1-\beta(1-\alpha)}\int_{\overline{w}}^{B} (w' - \overline{w})dF(w') \tag{26}$$

The left-hand side of this equation is the cost of choosing to continue searching, and the right-hand side is the expected discounted benefit obtained therefrom. Using equation (26), we can analyze the effects of changing the unemployment benefit c, the condition that F is in MPS, and changing the unemployment rate α. Here, these simulations are performed based on the following utility functions.

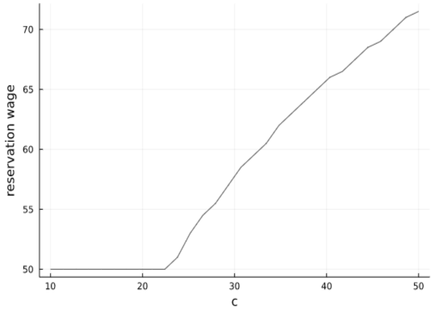


Fig.9: Unemployment benefits and reservation wages

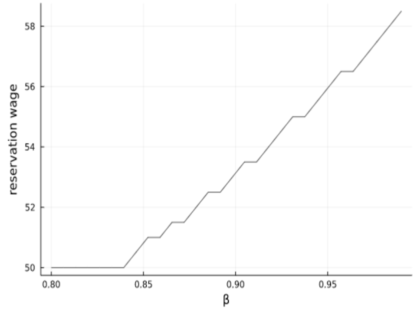


Fig.10: Discount factors and reservation wages

$$u(x) = \frac{x^{1-\sigma} - 1}{1 - \sigma}, \quad x = w, c$$

Additionally, suppose that an unemployed worker accepts a new wage offer with probability γ. If the worker accepts the offer, wages are drawn from the wage distribution F as before. Finally, the overall lifetime value of currently unemployed workers is:

$$U = c + \beta(1 - \gamma)U + \beta\gamma \int \hat{v}(w')dF(w')$$

In other words, the proportion 1 - γ that does not accept new wage offers will continue to be unemployed again. On the other hand, the rate γ of accepting a new offer estimates the lifetime value w'.

Here, we set β = 0.98, γ = 0.7, and σ = 2.0 to simulate. The wage distribution is the beta-binomial distribution, and the unemployment benefit c = 30.0 assumed in the basic McCall model. Furthermore, the separation rate is assumed to be α = 0.2. Examining Fig.9 and 10, it will be seen that the same results as the basic McCall model are obtained. In other words, higher unemployment benefits and discount factors lead to higher reservation wages, even after accounting for job separation and reemployment.

Next, let us confirm the effect on reservation wages when the separation rate changes. First, from the model, I calculate the effect of changes in α on the reservation wage $\overline{w}$:

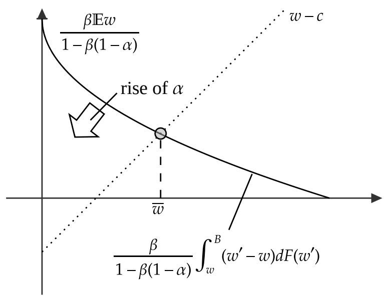


Fig.11: Separation rate and reservation wages

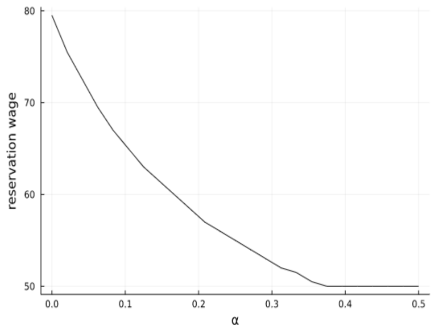


Fig.12: Simulation result

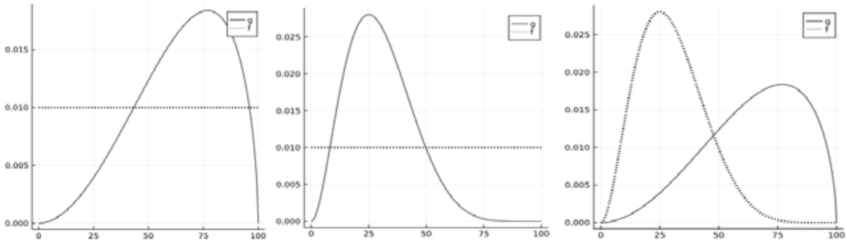
$$\overline{w}'(\alpha) = -\frac{\beta^2 \int_{\overline{w}(\alpha)}^{B} (w' - \overline{w}(\alpha))dF(w')}{[1-\beta(1-\alpha)]^2 + \beta[(1-(1-\alpha)][1-F(\overline{w}(\alpha))]} < 0$$

Therefore, an increase in the turnover rate lowers not only the expected utility of unemployed workers but also that of employed workers. This means that the solid line in Fig. 11 shifts downward. Fig.12 depicts the results of a simulation of the relationship between the separation rate α and the reservation wage $\overline{w}$.

In Fig.11, when the separation rate α rises, the solid line shifts downward and the reservation wage $\overline{w}$ declines. A simulation using default parameters is depicted in Fig.12, and it is clear that the reservation wage decreases as α increases from 0.0 to 0.5.

5. Learning Wage Distribution

So far, we have assumed that there is only one wage distribution. Here, we consider the case where there are two candidates for the density function of the wage distribution. One is F (density function is f) and the other is G (density function is g). For simplicity, we assume that job seekers know about these two distributions. At the beginning of the period, job seekers choose the sequences of wage $\{W_t\}$ to be either f or g. If $\pi 0$ is the (initial) probability of choosing f, the choice problem at time t is $\pi_t f + (1 - \pi_t)g$. Job



(a) Case 1　　(b) Case 2　　(c) Case 3

Fig.13: Wage distributions

seekers update this selection probability π_t based on observed offers.

The selection probability π is updated as follows:

$$\pi_{t+1} = \frac{\pi_t f(w_{t+1})}{\pi_t f(w_{t+1}) + (1-\pi_t)g(w_{t+1})} \tag{27}$$

Finally, the McCall model value function can be rewritten as:

$$V(w,\pi) = \max\left\{\frac{w}{1-\beta}, c + \beta \in V(w',\pi')h_\pi(w')dw'\right\} \tag{28}$$

where, $\pi' = q(\mathrm{w}', \pi')$. It should be noted that the inference π is a state variable because it influences job seekers' perceptions of future reward probabilities.

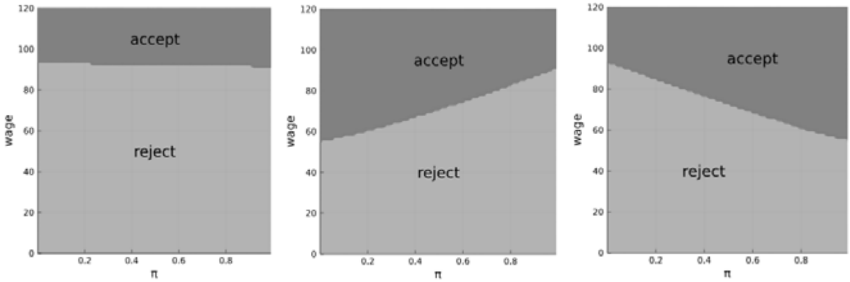
Here, we consider the following three cases by giving different hypotheses about f and g.

1. Case of flat perspective $f = B_n(1, 1)$ and optimistic perspective $g = B_n(3, 1.6)$.
2. Case of flat perspective $f = B_n(1, 1)$ and pessimistic perspective $g = B_n(3, 7)$.
3. Case of pessimistic perspective $f = B_n(3, 7)$ and optimistic perspective $g = B_n(3, 1.6)$.

These distribution functions are depicted in Fig.13.

A job seeker adjusts the selection probability π_t according to the observed offer, and selects (the density function of) the wage distribution f and g.

Next, a simulation analysis is performed. The $\pi \in (0.01, 0.99)$ and $w \in (0, 120)$ grids are set to 40, respectively. To solve this model, the Gauss-Legendre polynomials introduced in Section 2 are used for numerical integration. Furthermore, the approximation uses linear interpolation. The simulation results are depicted in Fig.14.



(a) Case 1 (b) Case 2 (c) Case 3

Fig.14: Wage distributions, selection probabilities, and wage

Now consider cases 1 and 2. In case 1, the uniform distribution is less attractive, and in case 2, the uniform distribution is more attractive. Also, depending on the value function, a policy that accepts an offer whenever w exceeds $\overline{w}$ and rejects it whenever w falls below wb is reasonable. Therefore, wb is the reservation wage, and the boundary line between accept and reject in Fig.14 corresponds to that level.

In Case 1, the uniform distribution *f* is an unattractive wage distribution, so an increase in π means that workers' perspective are less evaluated. As a result, the reservation wage will fall. In Case 2, the uniform distribution f is more attractive. Therefore, an increase in π means that the future prospects of workers will be evaluated higher. As a result, the reservation wage increases. Case 3 shows a more rapid decline in the reservation wage as the perspective shifts from optimistic to pessimistic.

6. Career selection

Neal (1999) has developed a search model that considers career matching in addition to employment matching (firm matching). The purpose of this model is to find reasons why young people change jobs or careers at an early stage, then seek jobs within a career, and eventually end up in a particular job.

Workers live indefinitely and get utility from career matching θ and firm matching ξ.

These are drawn from the distributions $F(\theta)$ and $G(\xi)$, respectively. At the beginning of each period, workers pick a new firm match ξ, or matching pair (θ, ξ). It should be noted here that workers can draw new careers only when drawing new firm matching. Therefore, a worker's choice is to remain in his/her current career and company, to transfer to a different company while continuing his/her previous career, or to join a new company under a new career. I also assume that workers do not look back because there is no learning or information transfer.

The worker's utility is given by $u_t = \xi_t + \theta_t$, as it comes from ξ_t and θ_t selected at the beginning of period t. Therefore, workers maximize $\mathbb{E}\Sigma_t(\xi_t + \theta_t)$. Here, if $v(\theta, \xi)$ is the optimal value of the worker problem at the beginning of the period, the Bellman equation is as follows:

$$v(\theta, \xi) = \theta + \xi + \beta \max \Big[v(\theta + \xi), \int v(\theta, \xi')g(\xi')d\xi', \iint v(\theta', \xi')f(\theta')g(\xi')d\theta' d\xi' \Big] \tag{29}$$

This maximization will be achieved at the beginning of the next period through following three action choices: (1) remain in θ and ξ, (2) keep career θ and extract new firm matching ξ, and (3) extract new career and new firm matching.

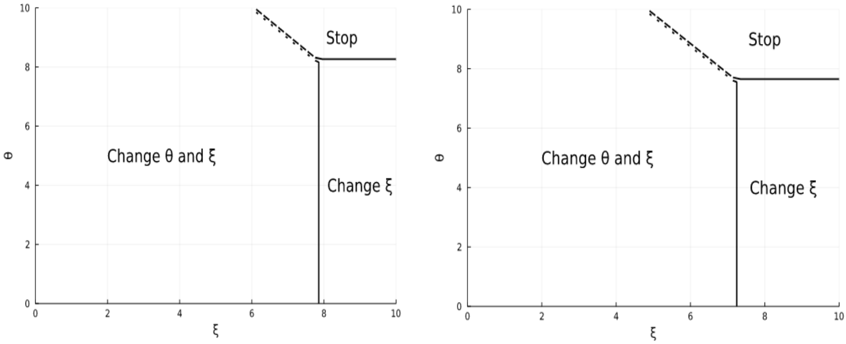
Carrying out a similar development to the process of deriving the reservation wage in the McCall model, we obtain the following:

$$\frac{\theta + \xi}{1 - \beta} \geq \max\{J(\xi), H\} \tag{30}$$

where, H is the value when drawing careers and firms (the second argument of the max function in Eq.(30)), and $J(\xi)$ is the value of drawing new firm matching while maintaining x. (the third argument of the max function in Eq.(30)). Eq.(30) states that staying in the current state (Neal defines this as "stop") is chosen when its value exceeds the value of the other two choices.

If the equation (30) holds with equality, the cutoff value $\overline{\theta}(\xi)$ is obtained as follows:

$$\overline{\theta}(\xi) = \max[(1 - \beta)J(\xi) - \xi, (1 - \beta)H - \xi]$$



(a) β = 0.98 (b) β = 0.90

Fig.15: Career selection and discount factors

If $\theta \geq \overline{\theta}(\xi)$, then the current state should be maintained. If this is not met, people will choose to move to other firms or start new careers.

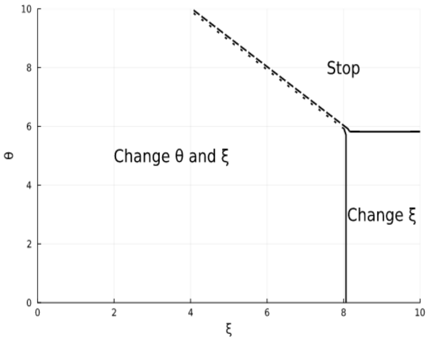
Also, the career cutoff value $\overline{\xi}$ is derived by solving:

$$J(\xi) = H$$

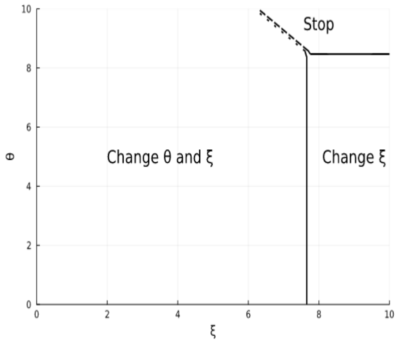
Based on this, if $\xi \geq \overline{\xi}$ or $\theta \geq \overline{\theta}$, workers will accept their current career.

Here is a simple simulation. First, for simplicity, assume that wage w_t is defined as the sum of earnings ξ_t and θ_t. I also assume that the distributions F and G follow the same beta binomial $B_n(1, 1)$ distribution in order to assume a flat situation with respect to careers and firms. Furthermore, the discount factor is assumed to be β = 0.98. In addition, we also examine the case of β = 0.90 to clarify how the selection results change depending on the discount factor. These results are depicted in Fig.15.

Fig.15 corresponds to Fig.1 of Neal (1999). Workers in region "Change θ and ξ" will choose a new pair (θ, ξ) in the next period, and workers in region "Change ξ" will choose a pair to maintain their careers and transfer firms. On the other hand, workers in "Stop" accept their current state. Fig.15(a) assumes that the discount factor is β = 0.98. In this case, workers try domain "Change θ and ξ" if they are unsatisfied with their current state, domain "Chnage ξ" if they are satisfied with their career but not with their company, and , would select the region "Stop" if they were satisfied with the current state. Also, Fig.12(b)



(a) $G = B_n(n, 100, 100)$



(b) $G = B_n(n, 0.8, 0.8)$

Fig.16: Career choice problem and career/firm distribution

depicts a case where the discount factor is $\beta = 0.90$. This means that the future is heavily discounted. As a result, workers are more likely to choose between "Stop" or "Change ξ".

Fig.16(a) shows a case where the wage distribution $G(\xi)$ for firm matching ξ is given by $B_n(n, 100, 100)$ (beta-binomial distribution). The distribution is nearly bell-shaped, meaning that the ξ-values are clustered around the mean. As a result, people are more likely to choose "Stop" because the chances of high-paying jobs appearing are low. On the other hand, Fig.16(b) is a case where $G(\xi)$ is $B(0.8, 0.8)$. Under this distribution, workers perceive that their current wages are lower than they could receive in new careers or other firms. Therefore, it is more likely that the area "Change θ and ξ" or "Change ξ" will be selected, and the worker will clearly try to change the current situation.

7. Coclusion

In this paper, I conducted a brief survey of dynamic programming solutions and simulation techniques. We also introduced the application of dynamic programming to the field of labor economics as an important exercise. Currently, with the evolution of free analysis software, it has become possible to perform extremely complex calculations and simulations. In particular, the importance of using such analytical methods is increasing in the construction and simulation of macroeconomic models.

Notes

1. See Puterman (2005) for Markov processes.
2. A metric space is a set X together with a function $d : X \times X \to \mathbb{R}$ that assigns a real number $d(x, y)$ to every pair $x, y \in X$

   satisfing the following properties:

   - $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$.
   - $d(x, y) = d(y, x)$.
   - $d(x, y) + d(y, z) \geq d(x, z)$.

   The last property is called the triangle inequality. A metric space is said to be complete if any Cauchy sequence of X converges to a point in X.
3. A Banach space is a complete normed space. A normed space defines a distance, and a vector space X on $\mathbb{C}$ is said to be normed if there is a defined function $\|\cdot\| : X \to \mathbb{R}$ called the norm satisfing (1) $\|x\| \geq 0$ $(x \in X)$, (2) $\|x\| \Leftrightarrow x = 0$, (3) $\|kx\| = |k|\,\|x\|$ $(x \in X, k \in \mathbb{C}$), and (4) $\|x + y\| \leq \|x\| + \|y\|$ $(x, y \in X$). A completeness is a space in which Cauchy sequences are always convergent sequences. A sequence on the normed space X is a Cauchy sequence if

   $$\lim_{n,m \to \infty} ||x_n - x_m|| = 0$$

   That is, for any $\varepsilon > 0$ there exists some $N \geq 1$ such that

   $$n, m \geq N \implies ||x_n - x_m|| < \varepsilon$$

   Here is an example of a simple Banach space. Let $C[a, b]$ be the vector space constituting a continuous function on $[a, b]$. If $\|f\| = \sup_{a \leq x \leq b} |f(x)|$ for $f \in C[a, b]$, then $C[a, b]$ is a Banach space. A proof of this is given below.

   Let $\{f_n\} \subset C[a, b]$ be a Cauchy sequence. Then for any $\varepsilon > 0$ there exists some $N \geq 1$, and for $n, m \geq N$, I can have $\|f_n - f_m\| \leq \varepsilon$. Therefore,

   $$|f_n(x) - f_m(x)| \leq \varepsilon, \quad a \leq x \leq b.$$

   If $f(x)$ is $f : [a, b] \to \mathbb{C}$, the sequence of functions $\{f_n\}$ converges to the function f at each point. Furthermore, in the above equation, as $m \to \infty$,

$$|f_n(x) - f(x)| \leq \varepsilon, \quad a \leq x \leq b$$

can be obtained. This means that $\{f_n\}$ uniformly converges to f and is $\lim_{n \to \infty} \|f_n - f\| \to 0$. Continuous functions uniformly converge to continuous functions, so $f \in C[a, b]$. Then $C[a, b]$ is complete because the Cauchy sequence $\{f_n\}$ has a destination f in $C[a, b]$.

4. Monotonicity means that if $v(s) \geq q(s)$ for all $s \in S$, then $\Gamma(v)(s) \geq \Gamma(q)(s)$ for all $s \in S$. See Add and Cooper (///) and Stokey, Lucas, and Prescott (1989) for details.
5. Discounting means that if we add a constant to v, the increase in $T(w)$ will be less than this constant. That is, $\Gamma(v + k)(s) \leq \Gamma(v)(s) + \beta k$ for some constant k and all $s \in S$.
6. This proof is performed using the triangle inequality.

$$\begin{aligned} ||v_N - v|| &= ||\Gamma_N(v_N) - \Gamma_N(v) + \Gamma_N(v) - \Gamma(v)|| \\ &\leq ||\Gamma_N(v_N) - \Gamma_N(v)|| + ||\Gamma_N(v) - \Gamma(v)|| \\ &\leq \beta||v_N - v|| + ||\Gamma_N(v) - \Gamma(v)|| \end{aligned}$$

7. Judd (///) indicates the case where the interpolation does not converge.
8. See Miao (///) for more details.
9. The double angle formula for cos θ shows that cos 2θ is a quadratic formula for cos θ, and the triple angle formula shows that cos 3θ is a cubic formula for cos θ.
10. Since $x = \cos\theta$, $\theta = \cos^{-1}x$. It is obtained by substituting this into $T_j(\cos\theta) = \cos j\theta$
11. This is the same as the Fourier series expansion of the periodic function $\hat{v}_j(x)$. Chebyshev polynomials drop the sine function part of the Fourier series expansion and perform variable transformation such as $x = \cos\theta$. The Chebyshev polynomial is the convergence theorem of the Fourier series expansion that can be applied to aperiodic functions.
12. See Judd (1998) for spline interpolation.
13. Now consider a spline interpolation for three points (x_0, y_0), (x_1, y_1), and (x_2, y_2). Therefore, the interval $[x_0, x_1]$ is interpolated with polynomial $S_0(x)$, and the interval $[x_1, x_2]$ is interpolated with polynomial $S_1(x)$. $S_0(x)$ passes through points (x_0, y_0) and (x_1, y_1) and $S_1(x)$ passes through points (x_1, y_1) and (x_2, y_2). Therefore, $S_0(x_0) = y_0$, $S_0(x_1) = y_1$, $S_1(x_1) = y_1$, and $S_1(x_2) = y_2$.

   Next, in order to connect $S_0(x)$ and $S_1(x)$ smoothly, it is necessary that the first derivative and the second derivative match at the connection point (x_1, y_1). Therefore, it is necessary that S_0' $(x_1) = S_1'$ (x_1) and S_0'' $(x_1) = S_1''$ (x_1).

Furthermore, as a condition, the second derivative at the end points is assumed to be zero, that is, S_0'' $(x_0) = 0$ and S_1'' $(x_2) = 0$. Since there are eight such constraints, eight unknowns can be solved.

14. See Judd (1998), Miranda and Fackler (2002), and Add and Cooper (2003) for more details.
15. If the integration interval is $[a, b]$, the following conversion will result in the [-1, 1] interval.

$$y = \frac{2x - a - b}{b - a}$$

16. A Legendre polynomial is an n-order polynomial that satisfies certain conditions regarding (1) recurrence, (2) Rodriguez formula, (3) generating function, and (4) orthogonality. According to the conditions of the recurrence formula, $P_0(x) = 1$ and $P_1(x)$, and $(n + 1)P_n+1(x) = (2n + 1)$ $xP_n(x)$ - $nP_{n\text{-}1}(x)$ for $n \geq 1$.
17. Now consider the case where $n = 2$. By recurrence condition and iterative substitution of Legendre polynomials, we can obtain

$$2P_2(X) = 3x^2 - 1.$$

The solution for $P_2(x) = 0$ is $x_i = \pm 1 / \sqrt{3}$, and substituting this into Eq. (14) yields $w_1 = w_2$ $= 1$. So the Gaussian quadrature is

$$\int_{-1}^{1} f(x)dx = 1f\left(-1/\sqrt{3}\right) + 1f\left(1/\sqrt{3}\right).$$

This approximation is exact if f is less than or equal to cubic. See Judd for more details.

18. Laffont (1990), Levy (1998), and Gollier (2001) use MPS for economic analysis.
19. This can be derived using integration by parts as follows:

$$\int_0^B [1 - F(w)]dw = \{p[1 - F(w)]\}_0^B + \int_0^B wdF(w)$$

Assuming F=0 and F=1, this means that

$$\int_0^B wdF(w) = \int_0^B [1 - F(w)]dw.$$

The left side of this equation defines $\mathbb{E}(W)$.

20. The single crossing property (SCP) is essential to the concept of mean preservation. The cumulative distribution function $G(\cdot)$ has a single crossing with $F(\cdot)$ if, for some value x, $G(x) \geq F(x)$ when $x > \bar{x}$ and $G(x) \leq F(x)$ when $x > \bar{x}$. See Diamond and Stiglitz (1974) and Machina and Pratt (1997) for the relationship between SCP and MPS.

21. Since this model assumes no recalls and there is no concept of turnover, workers who accept the offer get the present value of the wage stream shown below.

$$w + \beta w + \beta^2 w + \cdots = \frac{w}{1-\beta}$$

22. The first equation of equation (20) is decomposed into the interval $(0, \overline{w})$ and $(\overline{w}, B)$ as follows:

$$\frac{\overline{w}}{1-\beta}\int_0^{\overline{w}} dF(w') + \frac{\overline{w}}{1-\beta}\int_{\overline{w}}^{B} dF(w') = c + \beta\int_0^{\overline{w}} \frac{\overline{w}}{1-\beta} dF(w') + \beta\int_{\overline{w}}^{B} \frac{w'}{1-\beta} dF(w')$$

Here, c is used as it is because of the following.

$$\int_0^{B} dF(w') = \int_0^{\overline{w}} dF(w') + \int_{\overline{w}}^{B} dF(w') = 1$$

Summarizing the above equation by integration, we get

$$\frac{(1-\beta)\overline{w}}{1-\beta}\int_0^{\overline{w}} dF(w') - c = \beta\int_{\overline{w}}^{B} \frac{w'}{1-\beta} dF(w') - \frac{\overline{w}}{1-\beta}\int_{\overline{w}}^{B} dF(w'),$$

which is expressed as follows.

$$\overline{w}\int_0^{\overline{w}} dF(w') - c = \frac{1}{1-\beta}\int_{\overline{w}}^{B} (\beta w' - \overline{w}) dF(w')$$

Equation (21) can be obtained by adding the following to this:

$$\overline{w}\int_{\overline{w}}^{B} dF(w') = \frac{(1-\beta)\overline{w}}{1-\beta}\int_{\overline{w}}^{B} dF(w')$$

23. Add and subtract the following from the right side of equation (21).

$$\frac{\beta}{1-\beta}\int_0^{\overline{w}} (w' - \overline{w}) dF(w')$$

As a result, Equation (21) can be rewritten as follows:

$$\overline{w} - c = \frac{\beta}{1-\beta}\int_0^B (w' - \overline{w})dF(w') - \frac{\beta}{1-\beta}\int_0^{\overline{w}} (w' - \overline{w})dF(w').$$

Expanding the first term on the right side of this equation yields $\beta(\mathbb{E}w - \overline{w}) / (1 - \beta)$, so this equation can be rewritten as follows:

$$\overline{w} - (1-\beta)c = \beta\mathbb{E}w - \beta\int_0^{\overline{w}} (w' - \overline{w})dF(w').$$

On the other hand, if we apply integration by parts to the second term,

$$\beta\int_0^{\overline{w}} (w' - \overline{w})dF(w') = \beta(w' - \overline{w})F(w')\Big|_0^{\overline{w}} - \beta\int_0^{\overline{w}} F(w')dw'$$

and I have

$$\beta\int_0^{\overline{w}} (w' - \overline{w})dF(w') = -\beta\int_0^{\overline{w}} F(w')dw'.$$

Therefore,

$$\beta\int_0^{\overline{w}} (w' - \overline{w})dF(w') = \beta\int_0^{\overline{w}} F(w')dw'$$

and the expression (21) is obtained by expanding this.

24. Expand as in footnote 22. First, the formula for determining the reservation wage is as follows.

$$\frac{\overline{w}}{1-\beta} = c + \beta\int_0^{\overline{w}} \frac{\overline{w}}{1-\beta}dF(w') + \beta\int_{\overline{w}}^B \frac{w' + \beta\alpha\left(\frac{\overline{w}}{1-\beta}\right)}{1-\beta(1-\alpha)}dF(w')$$

Expanding out again in the same way as in footnote 22, I have:

$$\frac{\overline{w}}{1-\beta}\int_0^{\overline{w}} dF(w') + \frac{\overline{w}}{1-\beta}\int_{\overline{w}}^B dF(w') = c + \beta\int_0^{\overline{w}} \frac{\overline{w}}{1-\beta}dF(w') + \beta\int_{\overline{w}}^B \frac{w' + \beta\alpha\left(\frac{\overline{w}}{1-\beta}\right)}{1-\beta(1-\alpha)}dF(w')$$

Summarizing in the interval of integration, we obtain the following:

$$\overline{w}\int_{0}^{\overline{w}} dF(w') = c + \int_{\overline{w}}^{B}\left[\frac{\beta w' + \beta^2\alpha\left(\frac{\overline{w}}{1-\beta}\right)}{1-\beta(1-\alpha)} - \frac{\overline{w}}{1-\beta}\right]dF(w')$$

Furthermore, the second term on the left side of the following equation is added to both sides.

$$\overline{w}\int_{0}^{\overline{w}} dF(w') + \overline{w}\int_{\overline{w}}^{B} dF(w') = c + \int_{\overline{w}}^{B}\left[\frac{\beta w' + \beta^2\alpha\left(\frac{\overline{w}}{1-\beta}\right)}{1-\beta(1-\alpha)} - \frac{\overline{w}}{1-\beta} + \overline{w}\right]dF(w')$$

It is clear that the left-hand side of this equation is $\overline{w}$. By expanding the second term on the right side, we can finally obtain Equation (26).

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