C^∞ Reeb links have all Alexander polynomials

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Introduction

A C^r Reeb link is a link in S^3 which consists of the cores of all Reeb components of a codimension-one C^r foliation on S^3 (cf. [My]). In [G], Gabai showed that a non-trivial link is C^0 Reeb link iff it is non-splittable and also that so many knots are C^∞ Reeb knots. It is still open problem, however, whether every non-splittable (and non-trivial) link is a C^∞ Reeb link or not. Therefore there may exist a necessary condition for a link to be a C^∞ Reeb link.

In this note, we show that C^∞ Reeb links have all Alexander polynomials by a simple construction. More precisely, we show the following:

Theorem *For any link, there is a C^∞ Reeb link whose Alexander polynomial is the same one of the given link.*

We work in the smooth (C^∞) category. All foliations we consider will be smooth, of codimension-one.

1 A product formula of Alexander polynomials

Let V be an unknotted solid torus in S^3 with specified longitude and meridian and $L_0 = \bigcup_{i=1}^{n} K_i$ a link in V. Let K be a knot in S^3 and $\phi : V \to N(K)$ a diffeomorphism onto a tubular neighborhood $N(K)$ of K in S^3 which preserves the longitudes and the meridians. Then we define a new link L to be $\phi(L_0)$. The link type of L does not depend on the choice of ϕ and $N(K)$, and we call L the *satellite link* of K of type (V, L_0). Suppose that λ_i is the linking number of the meridian of V and $K_i (i = 1, \cdots, n)$. Then we have a product formula of the Alexander polynomials $\Delta(L_0; t_1, t_2,$

$\cdots, t_n)$, $\Delta(K;t)$, and $\Delta(L;t_1, t_2,\cdots, t_n)$, of L_0, K and L respectively:

Theorem (Torres [T]) *The following formula holds:*

$$\Delta(L;t_1, t_2,\cdots, t_n), = \Delta(K;t_1^{\lambda_1}t_2^{\lambda_2}\cdots t_n^{\lambda_n})\Delta(L_0;t_1, t_2,\cdots, t_n),$$

where $\Delta(K;t_1^{\lambda_1}t_2^{\lambda_2}\cdots t_n^{\lambda_n})$ *is the polynomial obtained by substituting* $t_1^{\lambda_1}t_2^{\lambda_2}\cdots t_n^{\lambda_n}$ *for* t *in the polynomial* $\Delta(K;t)$.

For the definition and fundamental facts about Alexander polynomials of links, we refer the reader to [T] and [R].

2 Construction and Proof

Suppose that an n-components link L_0 is given. Then, by a theorem of Alexander [A], we may consider L_0 as a closed braid. Precisely, L_0 is isotopic to link in a standard solid torus V in S^3 which is transverse to each meridian disk of V. Therefore we consider L_0 as such a link by an ambient isotopy of S^3. Let K_0 be any C^∞ Reeb knot. We can take a torus knot, for example, as K_0 (cf. [M], [My]). Let K be the untwisted double of K_0. Compare Figure 1 (for a precise definition, see [W] [R]).

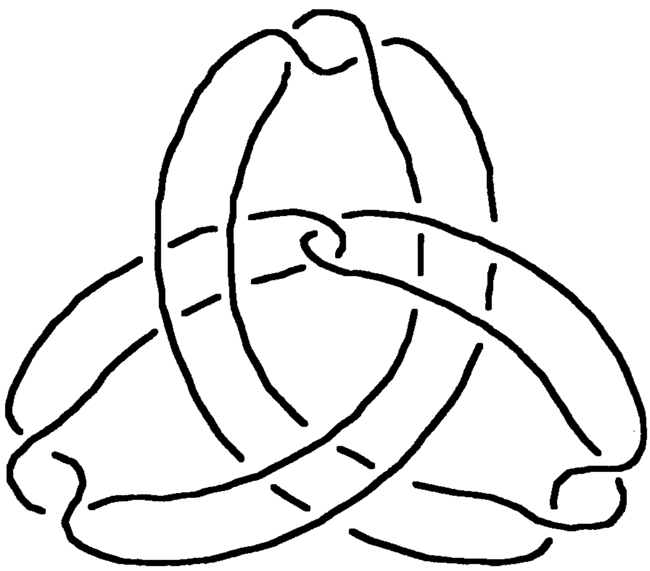

Figure 1. An untwisted double of a trefoil

We define a link L to be the satellite link of K of type (V, L_0). Then L is the desired link, that is, we have the following claims:

Claim 1 $\Delta(L;t_1, t_2,\cdots, t_n), = \Delta(L_0;t_1, t_2,\cdots, t_n)$,

Claim 2 *L is a C^∞ Reeb link.*

Proof of Claim 1 Since K is an untwisted double of K_0, $\Delta(K;t) = 1$ by an easy calculation (cf. [W], [R]). Then, apply Torres' theorem to L_0, K and L so as to obtain the identity. Q.E.D.

Proof of Claim 2 Suppose that K_0 is a torus knot. Then K_0 is fibred and therefore it is C^∞ Reeb knot by a standard construction, winding ends of fibres (cf. [M], [My]). We may assume that the boundary leaf of the Reeb component is a C^∞-flat leaf (i.e. the holonomy along the leaf is infinitely tangent to the identity). In the case that K_0 is an arbitrary C^∞ Reeb knot, the foliation can be approximated by such a foliation.

The exterior of K in S^3 is a compact 3-manifold obtained by attaching a Whitehead's link exterior to the exterior of K_0 (cf. Figure 2), where an attaching diffeomorphism takes a meridian and a longitude to a longitude and a meridian respectively.

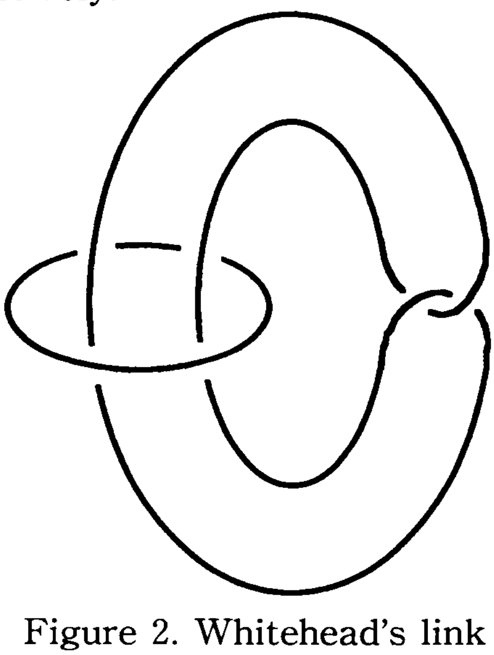

Figure 2. Whitehead's link

Since Whitehead's link is fibred (cf. [R], [Go]), it is easy to see that we can foliate V so that on a small tubular neiborhood of L_0 it induces the only Reeb component in V and ∂V is a leaf. It is enough for our claim that first we foliate V so that it is a Reeb component itself and then turbulize it along L_0 (see [My] for a precise construction). Finally, we attach them together so that we have a C^∞ foliation on S^3 whose only Reeb component is on a small tubular neiborhood of L. This completes the proof.

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