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The Sraffian System and Theories of Distribution and Effective Demand : Some Applications*

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I. Introduction

Without doubt, Piero Sraffa's slender book *Production of Commodities by Means of Commodities* (1960) had a great influence on the subsequent researches on the theories of capital and income distribution. In particular, Sraffa's theory of the 'standard commodity', which was proposed as a solution of the Ricardian puzzle in search for the 'invariant measure of value', offered an effective bridge between the highly aggregated Macroeconomic analysis and the disaggregated analysis of the economic interdependency. Although Sraffa's book is exclusively concerned with the static model, Goodwin (1983) suggested recently that Sraffian idea of the standard commodity displays its real ability when it is applied to the dynamic models. According to Goodwin, we can simplify the analysis of the complicated interdependent dynamics if we pay attention to the role of the standard commodity as an aggregator of the disaggregated dynamical system.

In this paper, we shall apply, following Goodwin (1983)'s suggestion, but probably more systematically than Goodwin (1983), the Sraffian idea to the analyses of the income distribution and the effective demand.

In section II, we formulate a general multisectoral model of production in an open economy with differential profit rates under the static setting, and reconsider the Sraffian proposition in our analytical framework.

In section III, we apply the Sraffian idea to a particular disaggregated dynamic model, i.e., the model of the wage-price spiral in an open economy. Mathematically, the original Sraffian standard commodity is defined as the right-Frobenius vector of the input coefficient matrix, but we shall show that we must re-define the matrix and we must apply the notion of the standard commodity to the newly defined matrix in order to use it as an aggregator of the particular dynamic system.

In section III, the dynamics of the price system is investigated and the hypothetical quantity system (the 'standard commodity') is used as an aggregator. On the other hand, we shall consider in section IV the 'dual' of the price system, the dynamics of the quantity system. In this section, we take up a multisectoral version of the Keynesian multiplier process, and propose the notion of the 'standard price', the dual notion of the standard commodity, as an aggregator of the disaggregated quantity dynamics. Section V is devoted to some concluding remarks. Finally, in the appendix we shall treat some Marxian themes which are not considered in the text.

II. The Basic Model

II-1. A One Sector Model of Production in an Open Economy

First of all, let us consider the very simple linear one sector model in a capitalist economy. Contrary to usual formulation, however, we shall introduce the international trade and the government explicitly into the picture keeping the structure of the model as simple as possible.

The price equation of such an economy may be formulated as

$$p = r(pa+qm)+(pa^{\ominus}+qm^{\ominus})+w\ell \qquad (1)$$

where the meanings of the symbols are as follows.

p = price level of the domestic product. q = price level of the imported mean of production in terms of the domestic currency. r = pre tax rate of profit. w = pre tax money wage rate. a = capital input coefficient of the domestic mean of production ($a > 0$). m

=capital input coefficient of the imported mean of production ($m>0$). $a^{\ominus}\equiv\delta a$=depreciation coefficient of the domestic mean of production ($0\leqq\delta\leqq1$). $m^{\ominus}\equiv d\,m$=depreciation coefficient of the imported mean of production ($0\leqq d\leqq1$). ℓ=labor input coefficient ($\ell>0$).

We can rewrite eq. (1) as

$$p=rp(a+\pi m)+p(a^{\ominus}+\pi m^{\ominus})+w\ell \tag{2}$$

where $\pi\equiv q/p$ is considered to be the *reciprocal* of the terms of trade if the domestic product is exportable. From this equation, we can easily derive the following relationship.

$$r=\{1/(a+\pi m)\}\{1-(a^{\ominus}+\pi m^{\ominus})-\omega\ell\} \tag{3}$$

where $\omega\equiv w/p$ is the real wage rate in terms of the domestic product. Eq. (3) expresses the pre tax wage-profit trade-off so that the potential conflict over income distribution between capital and labor in this simple economy(1).

Now, the share of pre tax wage in net national income is expressed as

$$\omega^{*}\equiv w\ell x/\{px-p(a^{\ominus}+\pi m^{\ominus})x\}\equiv\omega\ell/\{1-(a^{\ominus}+\pi m^{\ominus})\} \tag{4}$$

where x is the level of the domestic output.

From the equations (3) and (4) we have the following simple wage share-profit trade-off equation.

$$r=R(1-\omega^{*}) \tag{5}$$

where $R\equiv\{1-(a^{\ominus}+\pi m^{\ominus})\}/(a+\pi m)$ is the (pre tax) maximum rate of profit which is conditional not only on the technological parameters but also on the (reciprocal of the) terms of trade (π). We can easily see that $\partial R/\partial\pi<0$.

From eq. (5) we can derive the 'pre tax distribution frontier' as in Fig. 1. It is apparent that the position of the frontier is affected

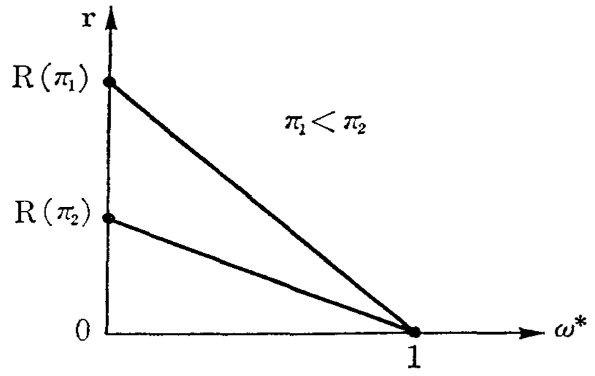

Fig. 1. Pre tax distribution frontier

by the terms of trade $(1/\pi)$. Namely, the improvement (the deterioration) of the terms of trade causes the upward (downward) shift of the pre tax distribution frontier(2).

Now, let us superimpose the effects of the taxation by the government on the above analysis. After tax rate of profit $(\tilde{r})$, after tax real wage rate $(\tilde{\omega})$ and after tax wage share $(\tilde{\omega}^*)$ are defined as follows respectivey.

$$\tilde{r} \equiv (1-\tau_r)r \tag{6}$$

$$\tilde{\omega} \equiv (1-\tau_w)\omega \tag{7}$$

$$\tilde{\omega}^* \equiv (1-\tau_w)\omega^* \tag{8}$$

where τ_r is the average tax rate on profit income and τ_w is the average tax rate on wage income $(0 \leqq \tau_r < 1$ and $0 \leqq \tau_w < 1)$(3). Substituting these relationships into the equations (3) and (5), we have

$$\tilde{r} = \{(1-\tau_r)/(a+\pi m)\}\{1-(a^{\ominus}+\pi m^{\ominus})-\tilde{\omega}\ell/(1-\tau_w)\} \tag{3$'$}$$

and

$$\tilde{r} = (1-\tau_r)R(\pi)\{1-\tilde{\omega}^*/(1-\tau_w)\} \tag{5$'$}$$

where $R(\pi) \equiv \{1-(a^{\ominus}+\pi m^{\ominus})\}/(a+\pi m) > 0$. From eq. (5)′ the after tax distribution frontier is derived (see Fig. 2)(4). It is apparent from this figure that the increase (the decrease) of the tax rates

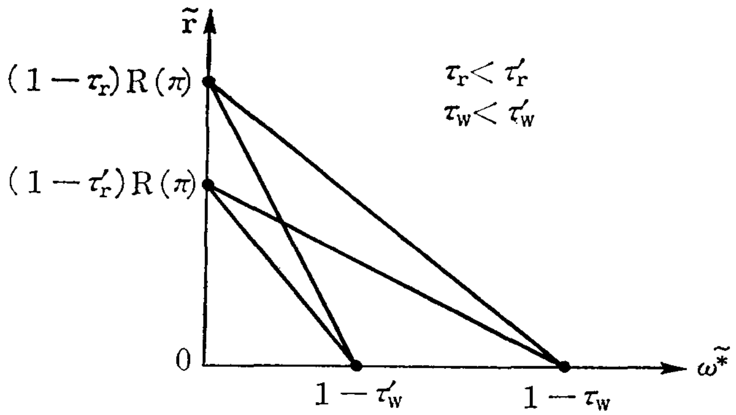

Fig. 2. After tax distribution frontier

will induce the downward (upward) shift of the after tax distribution frontier.

II-2. A General Multisectoral Model of Production in an Open Economy

The one sector model of production presented in section II-1 will turn out to be a powerful analytical tool for the analysis of the conflict over income distribution in an open capitalist economy for its simplicity and tractability if we can bridge a gap between the simple one sector world and the complex interdependent real world. It is well known that Sraffa (1960)'s ingeneous device of the so called 'standard commodity' provides such a bridge in case of the closed economy without government. In section II-3, we shall apply Sraffa's idea to the open economy with government. For this purpose, let us formulate the general multisectoral model of production in an open economy in this section. Although we abstract from the problems of the joint production and the choice of techniques, the results of the following analyses can be extended to the particular types of the joint production system (see Schefold (1978)).

The symbols which are used frequently throughout this paper are as follows.

a_{ij} =quantity of i'th domestic capital good which is used to produce a unit of j'th domestic good (i, j=1, 2, ……, n).

m_{hj} =quantity of h'th imported capital good which is used to produce a unit of j'th domestic good (h=1, 2, ……, s).

δ_{ij} =depreciation rate of i'th domestic capital good fixed in j'th domestic industry ($0 \leqq \delta_{ij} \leqq 1$).

d_{hj} =depreciation rate of h'th imported capital good fixed in j'th domestic industry ($0 \leqq d_{hj} \leqq 1$).

ℓ_j =quantity of direct labor input which is used to produce a unit of j'th domestic good.

p_j =price of j'th domestic good in terms of domestic currency.

q_h =price of h'th imported good in terms of domestic currency.

w =pre tax money wage rate in terms of domestic currency.

r_j =pre tax rate of profit in j'th domestic industry.

τ_w =average tax rate on wage income ($0 \leqq \tau_w < 1$).

τ_r =average tax rate on profit income ($0 \leqq \tau_r < 1$).

$$A \equiv \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{pmatrix} \geqq 0^{(5)}. \qquad A^{\ominus} \equiv \begin{pmatrix} \delta_{11}a_{11} & \delta_{12}a_{12} \cdots & \delta_{1n}a_{1n} \\ \delta_{21}a_{21} & \delta_{22}a_{22} \cdots & \delta_{2n}a_{2n} \\ \vdots & \vdots & \vdots \\ \delta_{n1}a_{n1} & \delta_{n2}a_{n2} \cdots & \delta_{nn}a_{nn} \end{pmatrix} \geqq 0.$$

$$M \equiv \begin{pmatrix} m_{11} & m_{12} \cdots & m_{1n} \\ m_{21} & m_{22} \cdots & m_{2n} \\ \vdots & \vdots & \vdots \\ m_{s1} & m_{s2} \cdots & m_{sn} \end{pmatrix} \geqq 0. \qquad M^{\ominus} \equiv \begin{pmatrix} d_{11}m_{11} & d_{12}m_{12} \cdots & d_{1n}m_{1n} \\ d_{21}m_{21} & d_{22}m_{22} \cdots & d_{2n}m_{2n} \\ \vdots & \vdots & \vdots \\ d_{s1}m_{s1} & d_{s2}m_{s2} \cdots & d_{sn}m_{sn} \end{pmatrix} \geqq 0.$$

$\ell \equiv [\ell_1, \ell_2, \cdots\cdots, \ell_n] > 0$. $\quad q \equiv [q_1, q_2, \cdots\cdots, q_s] > 0$.

$p \equiv [p_1, p_2, \cdots\cdots, p_n]$.

Now, let us assume as follows.

Assumption 1. The matrices A, $A^{\ominus}$, M, $M^{\ominus}$ and the vector ℓ are technologically determined so that they are constant.

Assumption 2. The matrix $A^{\ominus}$ is indecomposable(6).

Assumption 3. Each row of the matrix M is semipositive. In other words, each imported capital good is used in at least one domestic industry.

Assumption 4. Wages are paid out of current revenue rather than out of capital.

Then, the price system in our model can be expressed as follows.

$$p = (pA + qM)[\hat{r}] + pA^{\ominus} + qM^{\ominus} + w\ell \tag{9}$$

where $[\hat{r}]$ is the diagonal matrix of the pre tax profit rates, i.e., (7)

$$[\hat{r}] \equiv \begin{bmatrix} r_1 & & & \\ & r_2 & & 0 \\ & & \ddots & \\ & 0 & & r_n \end{bmatrix} \tag{10}$$

Now, the *reciprocal* of the terms of trade in terms of h'th import good is defined as

$$\pi_h \equiv q_h / (pf) \tag{11}$$

where $f = [f_1, f_2, \cdots f_n]' \geq 0$ is the unit vector of the export goods(8).

Then, we can rerwite eq. (9) as follows.

$$p = pC(\pi)[\hat{r}] + pC^{\ominus}(\pi) + w\ell \tag{12}$$

$$; \; C(\pi) \equiv A + f\pi M, \quad C^{\ominus}(\pi) \equiv A^{\ominus} + f\pi M^{\ominus}$$

where $\pi \equiv [\pi_1, \pi_2, \cdots\cdots, \pi_s]$.

Eq. (12) is an expression of the production system in an open economy in the 'closed' form(9). Henceforth, we treat the vector π as the positive parameter, but, it must be noted that π depends on the rate of foreign exchange as well as the price levels in the foreign country (see the footnote (2)).

Next, the pre tax real wage rate (ω) in terms of the commodity basket which the workers actually consume can be expressed as

$$\omega \equiv w/(pb^d + \tilde{q}b^f) \; ; \; \tilde{q} \equiv [q_1, q_2, \cdots\cdots, q_z] \tag{13}$$

where $b^d \equiv [b_1^d, b_2^d, \cdots\cdots, b_n^d]' \geq 0$ is the unit basket of the domestically produced wage goods and $b^f \equiv [b_1^f, b_2^f, \cdots\cdots, b_z^f]' \geqq 0$ is the unit basket of the imported wage goods. Substituting eq. (11) into eq. (13), we have

$$\omega = w/p(b^d + f\tilde{\pi}b^f) \tag{14}$$

where $\tilde{\pi} \equiv [\pi_1, \pi_2, \cdots\cdots, \pi_z]$.

Substituting eq. (14) into eq. (12), we obtain the following expression.

$$p[I - G(\hat{r}, \omega \; ; \; \pi, \tilde{\pi})] = 0 \tag{15}$$

where

$$\begin{aligned} G(\hat{r}, \omega; \pi, \tilde{\pi}) &\equiv C(\pi)[\hat{r}] + C^{\ominus}(\pi) + \omega(b^d + f\tilde{\pi}b^f)\ell \\ &\equiv (A + f\pi M)[\hat{r}] + (A^{\ominus} + f\pi M^{\ominus}) \\ &\quad + \omega(b^d + f\tilde{\pi}b^f)\ell. \end{aligned} \tag{16}$$

We shall confine the analysis to the case where all of r_is and ω are nonnegative. In this case, it follows from *Assumption 2* that G becomes to be an indecomposable nonnegative matrix. Then, eq, (15) implies that the relationship between the pre tax profit rates and the pre tax real wage rate is constrained by the following equation.

$$\lambda_G(r_1, \cdots\cdots, r_n, \omega : \pi, \tilde{\pi}) = 1 \tag{17}$$

where $\lambda_G(\cdot)$ is the Frobenius root of the matrix G. Furthermore, the price vector of the domestic products (p) is expressed as the left-Frobenius vector of the matrix G(10).

To assure the existence of the economically meaningful solutions, we must assume that

Assumption 5. The matrix $C^{\ominus}(\pi) = A^{\ominus} + f\pi M^{\ominus}$ is productive, i. e.,

$$\left\{ x \epsilon R^n_{++} \mid x > C^{\ominus}(\pi)x \right\} \neq \phi.$$

Under this assumption, we have $0 < \lambda_G(0, \cdots\cdots, 0, 0 ; \pi, \tilde{\pi}) = \lambda_C{}^{\ominus} < 1$, where $\lambda_C{}^{\ominus}$ is the Frobenius root of the matrix $C^{\ominus}$. Moreover, λ_G is the strictly increasing continuous function of r_is and ω because of the Perron-Frobenius theorem, and $\lambda_G > 1$ for sufficiently large r_i or ω. Therefore, we can determine the 'maximum pre tax rate of profit' in the i'th industry (r_i^{max}) and 'maximum pre tax real wage rate' (ω_{max}) *uniquely* as follows.

$$r_i^{max} \equiv \{ r_i > 0 \mid \lambda_G(0, \cdots\cdots, r_i, \cdots\cdots, 0, 0 ; \pi, \tilde{\pi}) = 1 \}$$

$$(i = 1, 2, \cdots\cdots, n) \tag{18}$$

$$\omega_{max} \equiv \{ \omega > 0 \mid \lambda_G(0, \cdots\cdots, 0, \omega ; \pi, \tilde{\pi}) = 1 \} \tag{19}$$

Then, eq. (17) defines the (n+1) dimensional 'pre tax wage-profit surface' for given π and $\tilde{\pi}$ in the domain $0 \leqq r_i \leqq r_i^{max}$ (i = 1, 2, ……, n) and $0 \leqq \omega \leqq \omega_{max}$, in which the locus of any combination of two variables becomes a strictly decreasing continuous function for

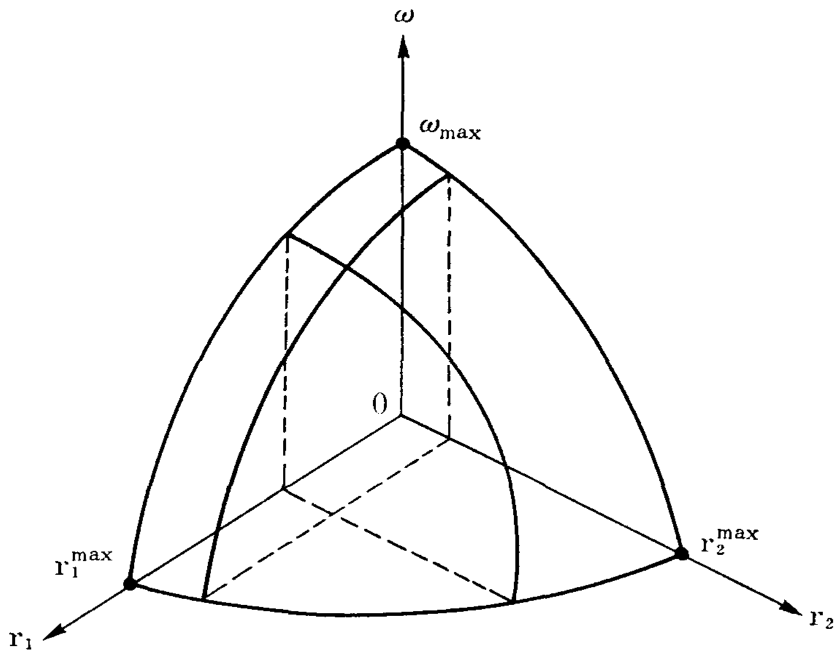

Fig. 3. Pre tax wage-profit surface in a two sector economy ; An example

given values of the remaining $(n-1)$ variables(11). An example of the pre tax wage-profit surface in case of the two sector economy is illustrated in Fig. 3.

By the way, the graph of the usual pre tax wage-profit curve with equal rate of profit is the monotonically decreasing continuous curve charactalized as

$$\Omega \equiv \{(r,\ \omega) \epsilon R_+^2 \mid \lambda_G(r,\ \cdots\cdots,\ r,\ \omega\ ;\ \pi,\ \tilde{\pi}) = 1\} \qquad (20)$$

, which is expressed as the projection of $\Delta_1 \cap \Delta_2$ to the arbitrary $r_i - \omega$ plane, where Δ_1 is the graph of the pre tax wage-profit surface and Δ_2 is the graph of the hyperplane characterized by $r_1 = r_2 = \cdots \cdots = r_n$ (see Fig. 4).

Note that the 'maximum pre tax equal rate of profit' (R) which is characterized as

$$R \equiv \{r > 0 \mid \lambda_G(r,\ \cdots\cdots,\ r,\ 0\ ;\ \pi,\ \tilde{\pi}) = 1\} \qquad (21)$$

is less than any of r_i^{max}, i. e.,

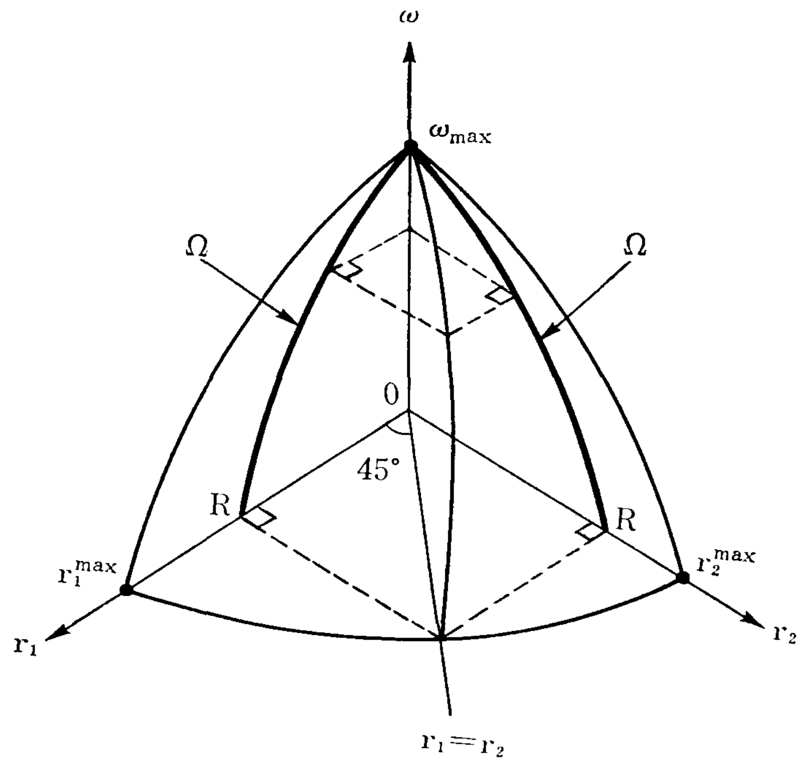

Fig. 4.

$$0 < R < \min\{r_1^{max},\ r_2^{max},\ \cdots\cdots,\ r_n^{max}\}. \tag{22}$$

Now, we can show, roughly speaking, that the inprovement (the deterioration) of the terms of trade causes the upward (downward) shift of the pre tax wage-profit surface. More accurately, we have the following

Theorem 1.

(1) Suppose that $r_i > 0$ for all i and that an *arbitrary* element of the vector π is increased (decreased). Then,

(i) ω must decrease (increase) if all the rates of profit are constant, and

(ii) r_i must decrease (increase) if ω and all the rates of profit other than r_i are constant.

(2) Suppose that $\omega > 0$ and that an element π_k of the vector $\tilde{\pi}$ such that $b_k^f > 0$ is increased (decreased). Then, the above statements (i) and (ii) follow.

(Proof.)

We shall only prove the proposition (1). In a similar way, the proposition (2) can be easily proved.

Suppose, without loss of generality, that $\Delta\pi_k > 0$, $\Delta\pi_q = 0$ $(q \neq k)$, $f_i > 0$ and $m_{kj} > 0$ (12). Then, we have

$$\Delta C = f(\Delta\pi)M$$

$$= \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \underbrace{[0, \cdots\cdots, \Delta\pi_k, 0, \cdots\cdots, 0]}_{s} \begin{pmatrix} m_{11} & m_{12} \cdots\cdots m_{1n} \\ m_{21} & m_{22} \cdots\cdots m_{2n} \\ \vdots & \vdots \quad\quad \vdots \\ m_{s1} & m_{s2} \cdots\cdots m_{sn} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 \cdots\cdots 0 & f_1\Delta\pi_k & 0 \cdots\cdots 0 \\ 0 \cdots\cdots 0 & f_2\Delta\pi_k & 0 \cdots\cdots 0 \\ \vdots \quad \vdots & \vdots & \vdots \quad \vdots \\ 0 \cdots\cdots 0 & f_n\Delta\pi_k & 0 \cdots\cdots 0 \end{pmatrix}}_{s} \begin{pmatrix} m_{11} \cdots\cdots m_{1j} \cdots\cdots m_{1n} \\ \vdots \quad \vdots \quad \vdots \\ m_{k1} \cdots\cdots m_{kj} \cdots\cdots m_{kn} \\ \vdots \quad \vdots \quad \vdots \\ m_{s1} \cdots\cdots m_{sj} \cdots\cdots m_{sn} \end{pmatrix}$$

$$= \begin{pmatrix} f_1\Delta\pi_k m_{k1} \cdots\cdots f_1\Delta\pi_k m_{kj} \cdots\cdots f_1\Delta\pi_k m_{kn} \\ f_2\Delta\pi_k m_{k1} \cdots\cdots f_2\Delta\pi_k m_{kj} \cdots\cdots f_2\Delta\pi_k m_{kn} \\ \vdots \quad \vdots \quad \vdots \\ f_i\Delta\pi_k m_{k1} \cdots\cdots f_i\Delta\pi_k m_{kj} \cdots\cdots fi\Delta\pi_k m_{kn} \\ \vdots \quad \vdots \quad \vdots \\ f_n\Delta\pi_k m_{k1} \cdots\cdots f_n\Delta\pi_k m_{kj} \cdots\cdots f_n\Delta\pi_k m_{kn} \end{pmatrix}.$$

Since $f_i\Delta\pi_k m_{kj} > 0$ by assumption, we have $\Delta C \geq 0$. If $r_i > 0$ for all $i \epsilon \{1, 2, \cdots\cdots, n\}$, the diagonal elements of the matrix

$[\hat{r}]$are positive so that we have $\Delta G = \Delta C[\hat{r}] \geq 0$ for given r_is. In this case, it follows from the Perron-Frobenius theorem that λ_G must *increase* for given r_is and ω. Therefore, at least one of the values of r_is or ω must *decrease* to restore the equality of eq. (17). It is obvious that the converse is true in the case of $\Delta\pi_k < 0$.

(q. e. d.)(13)

Now, let us introduce the effects of the taxation into the model. The after tax rate of profit ($\tilde{r}_i$), the after tax money wage rate ($\tilde{w}$) and the after tax real wage rate ($\tilde{\omega}$) are defined as follows.

$$\tilde{r}_i \equiv (1-\tau_r) r_i \quad (i = 1, 2, \cdots\cdots, n) \tag{23}$$

$$\tilde{w} \equiv (1-\tau_w) w \tag{24}$$

$$\tilde{\omega} \equiv (1-\tau_w) \omega \tag{25}$$

Substituting these relationships into the equations (12), (15) and (16), we have the following modified equations.

$$p = pC(\pi)[\hat{r}]\frac{1}{1-\tau_r} + pC^{\ominus}(\pi) + \frac{\tilde{w}}{1-\tau_w}\ell \tag{12$'$}$$

$$p[I - G(\hat{r},\ \tilde{\omega}\ ;\ \pi,\ \tilde{\pi},\ \tau_r,\ \tau_w)] = 0 \tag{15$'$}$$

$$\begin{aligned} &G(\hat{r},\ \tilde{\omega}\ ;\ \pi,\ \tilde{\pi},\ \tau_r,\ \tau_w) \\ &\equiv C(\pi)\ [\hat{r}]\frac{1}{1-\tau_r} + C^{\ominus}(\pi) + \frac{\tilde{\omega}}{1-\tau_w}(b^d + f\tilde{\pi}b^f)\ell \\ &\equiv (A + f\pi M)\ [\hat{r}]\frac{1}{1-\tau_r} + (A^{\ominus} + f\pi M^{\ominus}) + \frac{\tilde{\omega}}{1-\tau_w}(b^d + f\tilde{\pi}b^f)\ell \end{aligned} \tag{16$'$}$$

where

$$[\hat{r}] = \begin{bmatrix} \tilde{r}_1 & & & 0 \\ & \tilde{r}_2 & & \\ & & \ddots & \\ 0 & & & \tilde{r}_n \end{bmatrix}. \tag{26}$$

In this case, eq. (17) is also modified as

$$\lambda_G(\tilde{r}_1, \cdots\cdots, \tilde{r}_n, \tilde{\omega} \; ; \pi, \tilde{\pi}, \tau_r, \tau_w) = 1 \tag{17'}$$

which defines the after tax wage-profit surface. We can easily see that the increase (the decrease) of τ_r or τ_w causes the downward (upward) shift of the after tax wage-profit surface.

II-3. Standard System in an Open Economy

In the previous section, we managed to analyze the wage-profit trade off in an open economy under the multisectoral setting at the cost of the lucidity of the simple one sector model. However, an application of Sraffa (1960)'s idea of the 'standard commodity' to the present model will be helpful to restore the analytical lucidity. For this purpose, let us consider the Sraffian notion of the 'standard system' in the context of an open economy.

The 'standard system' is defined as an activity vector $x^*=[x_1^*, x_2^*, \cdots\cdots, x_n^*]'>0$ such that it assures "a uniform rate of surplus throughout economic system" (Pasinetti(1977) p. 96). In the context of the present model, we may say that a standard system exists if there exist a $(n \times 1)$ vector $x^*>0$ and a scalar $R>0$ which satisfy the following equation.

$$[I-C^{\ominus}(\pi)]x^*=RC(\pi)x^* \tag{27}$$

or equivalently,

$$[(1/R)I-[I-C^{\ominus}(\pi)]^{-1}C(\pi)]x^*=0 \tag{27'}$$

where R is the 'uniform rate of surplus' which is called the 'standard ratio'. *Assumption 5* implies that $[I-C^{\ominus}(\pi)]^{-1}=\sum_{t=0}^{\infty}\{C^{\ominus}(\pi)\}^t>0$, so that the matrix $[I-C^{\ominus}(\pi)]^{-1}C(\pi)$ becomes to be indecomposable from *Assumption 2*[14].

Therefore, the Perron-Frobenius theorem assures that the standard system exists, and x^* is determined uniquely up to scalar multiplication. Moreover, it is easily shown that the 'standard ratio' R coincides with the 'maximum pre tax equal rate of profit' which was defined in the previous section, and it is a continuous *decreasing* function of each element of the vector π, i. e.,

$$R = R(\pi) \equiv R(\underset{\ominus}{\pi_1}.\ \underset{\ominus}{\pi_2},\ \cdots\cdots,\ \underset{\ominus}{\pi_s}). \tag{28}$$

Now, Sraffian 'standard national income' (Y^*) in the present context is defined as

$$Y^* \equiv p[I - C^{\ominus}(\pi)]x^*. \tag{29}$$

Moreover, following Sraffa (1960), let us normalize the level of the activity vector x^* so as to satisfy the following condition.

$$\ell x^* = 1 \tag{30}$$

Sraffa (1960) proved, in the context of the model of the closed economy with equal rate of profit, that the lucidity of the simple one sector model is restored if the real wage rate is measured in terms of the (hypothetical) standard national income rather than in terms of the basket of the wage goods which the workers actually consume. Now, we shall confirm this Sraffian proposition in our framework of the model of an open economy with differential rates of profit.

First, let us define the pre tax real wage rate in terms of the standard national income (ω^*) as follows(15).

$$\omega^* \equiv w/(p[I - C^{\ominus}(\pi)]x^*) \tag{31}$$

Second, let ud define the 'average pre tax rate of profit' (r^*) as follows by using the standard activity vector (x^*) as weights.

$$r^* \equiv (px^* - pC^{\ominus}(\pi)x^* - w\ell x^*)/(pC(\pi)x^*) \tag{32}$$

Then, we have the following 'Sraffian equation' in our model.

Theorem 2.

$$r^* = R(\pi)(1-\omega^*) \tag{33}$$

(Proof.)

From the equations (12) and (30) we have

$$p[I-C^{\ominus}(\pi)]x^* - w = pC(\pi)[\hat{r}]x^*, \tag{34}$$

which implies from eq. (31) that

$$1-\omega^* = (pC(\pi)[\hat{r}]x^*)/(p[I-C^{\ominus}(\pi)x^*). \tag{35}$$

On the other hand, from eq. (27) we have

$$p[I-C^{\ominus}(\pi)]x^* = R\,pC(\pi)x^* \tag{36}$$

and it follows from eq. (12) and eq. (32) that

$$r^* = (pC(\pi)[\hat{r}]x^*)/(pC(\pi)x^*). \tag{37}$$

Substituting eq. (36) and eq. (37) into eq. (35), we obtain

$$1-\omega^* = r^*/R \tag{38}$$

which is the desired equation.

(q. e. d.)

Now, we can define the after tax real wage rate in terms of the standard national income ($\tilde{\omega}^*$) and the after tax average rate of profit ($\tilde{r}^*$) as follows.

$$\tilde{\omega}^* = (1-\tau_w)\omega^* \tag{39}$$

$$\tilde{r}^* \equiv (1-\tau_r)r^* \tag{40}$$

Substituting these relationships into eq. (33), we obtain

$$\tilde{r}^* = (1-\tau_r)R(\pi)\{1-\tilde{\omega}^*/(1-\tau_w)\}. \tag{33$'$}$$

Note that eq. (33) and eq. (33)′ are exactly coincide with eq. (5) and eq. (5)′ in a one sector model respectively. In other words, Sraffian standard commodity is a powerful 'aggregator' which assures that the simple one sector model is a correct 'surrogate' of the highly complex and interrelated real economic world.

III. An Application to Dynamic Analysis

III-1. Conflict over Income Distribution and the Wage-Price Spiral in an Open Economy

Goodwin (1983) suggested that Sraffa's device displays its real ability when it is applied to the dynamic model, i. e., it serves as a powerful aggregator of the disaggregated dynamical system. In this section, we shall apply Goodwin's idea to a particular type of the dynamic model, namely, a model of the wage-price spiral in an open economy. For this purpose, first, we shall consider a dynamic version of the simple one sector model.

The lagged mark-up process in a one sector setting may be formulated as follows by 'dynamizing' the model of the section II-1.

$$p_t = \left\{ \frac{\tilde{r}}{1-\tau_r}(a+\pi m) + (a^{\ominus}+\pi m^{\ominus}) \right\} p_{t-1} + w_t \ell \tag{41a}$$

$$w_t = \frac{\tilde{\omega}}{1-\tau_w} p_{t-1} \tag{41b}$$

where $\tilde{r}$ is the 'required' after tax rate of profit or the 'mark-up' which is set by the capitalists, and $\tilde{\omega}$ is the 'required' after tax real wage rate which is demanded by the workers. Eq. (41a) says that the capitalists set the price of the domestic good in the period t so as to satisfy their requiment on the basis of the production cost in the previous period. (It is assumed, however, that there is no time lag between the wage payment and the pricing.) Eq. (41b) implies that the workers require the money wage rate in the period t which satisfies their requirement on the basis of the price in the previous period, and their demand is accepted by the capitalists. (For simplicity, it is assumed in this model that the workers consume only the domestic good.)

Substituting eq. (41b) into eq. (41a), we have the following very simple difference equation.

$$p_t = \lambda(\tilde{r}, \tilde{\omega} ; \pi, \tau_r, \tau_w) p_{t-1} \tag{42}$$

where

$$\lambda(\tilde{r}, \tilde{\omega} ; \pi, \tau_r, \tau_w) \equiv \frac{\tilde{r}}{1-\tau_r}(a+\pi m) + (a^{\ominus}+\pi m^{\ominus}) + \frac{\tilde{\omega}}{1-\tau_w}\ell$$

$$; \partial\lambda/\partial\tilde{r} > 0, \ \partial\lambda/\partial\tilde{\omega} > 0, \ \partial\lambda/\partial\pi > 0, \ \partial\lambda/\partial\tau_r > 0, \ \partial\lambda/\partial\tau_w > 0. \tag{43}$$

From eq. (41b) and eq. (42) the rates of price inflation and money wage inflation are expressed as follows.

$$\Delta p/p \equiv (p_t - p_{t-1})/p_{t-1} = \lambda - 1 \tag{44a}$$

$$\Delta w/w \equiv (w_t - w_{t-1})/w_{t-1} = (p_{t-1} - p_{t-2})/p_{t-2} = \lambda - 1 \tag{44b}$$

The locus of $\tilde{r}$ and $\tilde{\omega}$ which satisfies $\lambda = 1$ coincides with the locus defined by eq. (3)′ in section II-1, which is nothing but the 'after tax wage-profit frontier'. We can see from eq. (44a) and (44b) that the rate of inflation becomes to be positive ($\lambda > 1$) if and only if the combination of $\tilde{r}$ and $\tilde{\omega}$ is 'above' the after-tax wage-profit frontier. In other words, the wage-price spiral occurs if and only if the requirements of the capitalists and the workers are inconsistent each other compared with the existing technological and economic conditions. On the other hand, we can define the 'realized' after tax rate of profit ($\tilde{r}_t^{**}$) and the 'realized' after tax real wage rate ($\tilde{\omega}_t^{**}$) as follows.

$$\tilde{r}_t^{**} \equiv \frac{(1-\tau_r)\{p_t-(a^{\ominus}+\pi m^{\ominus})p_t-w_t \ell\}}{p_t(a+\pi m)} \tag{45}$$

$$\tilde{\omega}_t^{**} \equiv (1-\tau_w)w_t/p_t \tag{46}$$

Substituting eq. (41b) and the relationship $p_t=\lambda p_{t-1}$ into eq. (41a) and eq. (45), we have

$$\tilde{r}_t^{**}=\frac{(1-\tau_r)\left\{1-(a^{\ominus}+\pi m^{\ominus})-\frac{1}{1-\tau_w}\frac{\tilde{\omega}}{\lambda}\ell\right\}}{(a+\pi m)}$$

$$<\frac{(1-\tau_r)\left\{1-(a^{\ominus}+\tau m^{\ominus})\frac{1}{\lambda}-\frac{1}{1-\tau_w}\frac{\tilde{\omega}}{\lambda}\ell\right\}}{(a+\pi m)\frac{1}{\lambda}}$$

$$=\tilde{r} \quad (\text{if } \lambda>1) \tag{47}$$

Next, substituting $p_t=\lambda p_{t-1}$ into eq. (41b) and comparing with eq. (46), we obtain

$$\tilde{\omega}_t^{**}=\tilde{\omega}/\lambda<\tilde{\omega} \quad (\text{if } \lambda>1). \tag{48}$$

It is easily seen that the combination of $\tilde{r}_t^{**}$ and $\tilde{\omega}_t^{**}$ is necessarily on the after tax wage-profit frontier (see Fig. 5). In other words,

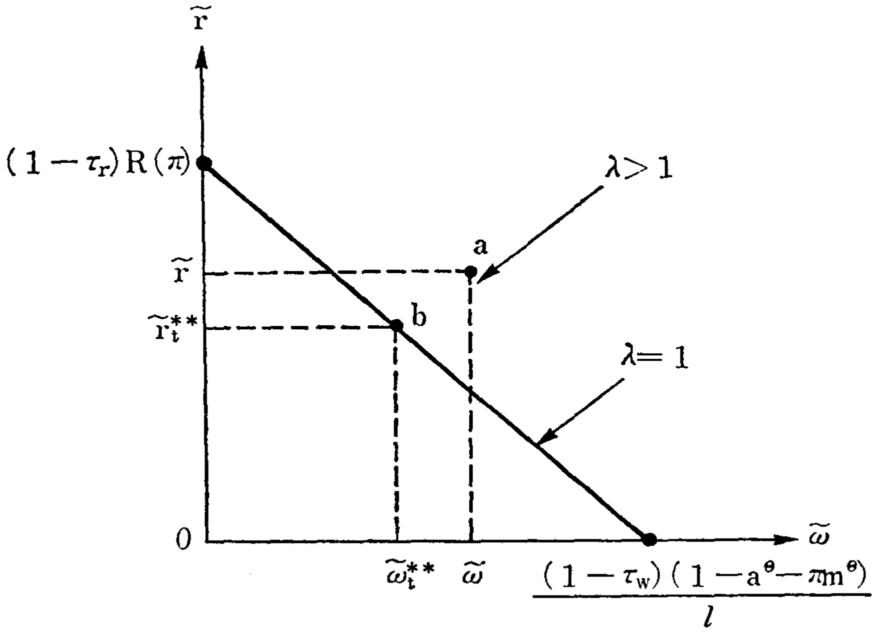

Fig. 5.

neither of the requiments of both classes is satisfied in the process of the wage-price spiral. It is also easily proved that (i) if $\tilde{r}$ is given, the larger $\tilde{\omega}$, the smaller is $\tilde{r}_t^{**}$ so that the larger is $\tilde{\omega}_t^{**}$, (ii) if $\tilde{\omega}$ is given, the larger $\tilde{r}$, the smaller is $\tilde{\omega}_t^{**}$ so that the larger is $\tilde{r}_t^{**}$, and (iii) the increase (decrease) of the tax rates or the deterioration (improvement) of the terms of trade causes the downward (upward) shift of the after tax wage-profit frontier so that it accelerates (decrlerates) inflation and lowers (raises) $\tilde{r}_t^{**}$ and $\tilde{\omega}_t^{**}$.

Now, let us reconsider the problem in a multisectoral setting. The multisectoral counterparts of eq. (41a) and eq. (41b) may be written as follows[16].

$$p(t)=p(t-1)\,[C(\pi)\,[\hat{r}]\frac{1}{1-\tau_r}+C^{\ominus}(\pi)]+w(t)\ell \qquad (41a)'$$

$$w(t)=p\,(t-1)\,(b^d+f\,\tilde{\pi}\,b^f)\frac{\tilde{\omega}}{1-\tau_w} \qquad (41b)'$$

where $p(t)\equiv[p_1(t),\ p_2(t),\ \cdots\cdots,\ p_n(t)]$, and other symbols are the same as those which are defined in section II-2.

Substituting eq. (41b)′ into eq. (41a)′ gives

$$p(t)=p(t-1)G(\hat{r},\ \tilde{\omega}\ ;\ \pi,\ \tilde{\pi},\ \tau_r,\ \tau_w) \qquad (49)$$

where the matrix $G\ (\cdot)$ is defined by eq. (16)′.

It is easy to write the formal solution of eq. (49), i. e.,

$$p(t)=p(0)G^t. \qquad (50)$$

The assumptions of section II-2 implies that G is an indecomposable nonnegative matrix[17]. Furthermore, in tshis section, we shall assume that

Assumption 6. The matrix G is primitive, i. e., there is *no* permutation matrix P which transforms G into

$$P^{-1}GP=\begin{pmatrix} 0 & 0 \cdots\cdots 0 & G_{1k} \\ G_{21} & 0 \cdots\cdots\cdots & 0 \\ 0 & G_{32} \ddots & \vdots \\ \vdots & \ddots \quad \ddots & \vdots \\ 0 \cdots\cdots & 0 \quad G_{kk-1} & 0 \end{pmatrix}$$

where G_{ij} is the nonnegative submatrix which is not necessarily square.

Note that a *sufficient* condition of the primitivity of the matrix G is that at least one diagonal element of G is positive, which is a reasonable condition.

It is well known that under *Assumption 6*, there exists $\lim_{t\to\infty}(G/\lambda_G)^t = B > 0$, where λ_G is the Frobenius root of the matrix G(18). Then, it follows that

$$\lim_{t\to\infty}(p(t)/\lambda_G^t) = \lim_{t\to\infty} p(0)\ (G/\lambda_G)^t = p(0)\ B > 0\,. \tag{51}$$

Therefore, from eq. (41b)′ and eq. (51) we have

$$\lim_{t\to\infty}(w(t)/\lambda_G^t) = \lim_{t\to\infty}\left(\frac{p(t-1)}{\lambda_G^{t-1}}\right)(b^d + f\tilde{\pi}b^f)\frac{\tilde{\omega}}{(1-\tau_w)\lambda_G}$$

$$= p(0)\ B(b^d + f\tilde{\pi}b^f)\frac{\tilde{\omega}}{(1-\tau_w)\lambda_G} > 0\,. \tag{52}$$

The equations (51) and (52) imply that

$$\lim_{t\to\infty}\frac{p_i(t)-p_i(t-1)}{p_i(t-1)} = \lim_{t\to\infty}\frac{w(t)-w(t-1)}{w(t-1)} = \lambda_G - 1$$

$$(i=1,\ 2,\ \cdots\cdots,\ n). \tag{53}$$

In other words, the relative prices and the real wage rate converge to the definite values ultimately.

Now, it is evident that the wage-price inflation occurs if and only if

$$\lambda_G(\tilde{r}_1,\ \cdots\cdots,\ \tilde{r}_n,\ \tilde{\omega}\ ;\ \pi,\ \tilde{\pi},\ \tau_r,\ \tau_w) > 1\,. \tag{54}$$

In other words, wage-price inflation occurs if and only if the combination of the required after tax profit rates and the required after tax real wage rate is 'above' the after tax wage-profit surface so that the distributive requirements of the economic agents are inconsistent each other.

The 'realized' after tax rate of profit in the j'th industry ($\tilde{r}_j^{**}(t)$) and the 'realized' after tax real wage rate ($\tilde{\omega}^{**}(t)$) can be defined as follows.

$$r_j^{**}(t) \equiv \frac{(1-\tau_r)\{p(t)I^{(j)} - p(t)C^{\ominus}(\pi)^{(j)} - w(t)\ell_j\}}{p(t)C(\pi)^{(j)}} \tag{55}$$

$$\tilde{\omega}^{**}(t) \equiv \frac{(1-\tau_w)\,w(t)}{p(t)(b^a + f\bar{\pi}b^f)} \tag{56}$$

where $X^{(j)}$ is the j'th column of the matrix X. It is easily seen that the combination ($\tilde{r}_1^{**}$, $\tilde{r}_2^{**}$, ……, $\tilde{r}_n^{**}$, $\tilde{\omega}^{**}$) lies on the after tax wage-profit surface. Then, we can prove the following two theorems concerning the 'comparative dynamics'.

Theorem 3.

Suppose that $\lambda_G > 1$. Then, it follows that $\lim_{t\to\infty} \tilde{r}_j^{**}(t) < \tilde{r}_j$ for all $j \in \{1, 2, \cdots\cdots, n\}$ and $\lim_{t\to\infty} \tilde{\omega}^{**}(t) < \tilde{\omega}$.

(Proof.)

(i) From the equations (41a)′, (41b)′, (51), (52), (55) and (56), we have the following relationships since $\tilde{r}_j$ is invariant through the dynamical adjustment process.

$\lim_{t\to\infty} \tilde{r}_j^{**}(t)$

$$=\lim_{t\to\infty}\frac{(1-\tau_r)\left\{p(t)/\lambda_G^t\right\}[I^{(j)}-C^{\ominus}(\pi)^{(j)}-(b^d+f\tilde{\pi}b^f)\frac{\tilde{\omega}\ell_j}{(1-\tau_w)\lambda_G}]}{\left\{p(t)/\lambda_G^t\right\}C(\pi)^{(j)}}$$

$$=\frac{(1-\tau_r)\{p(0)B\}[I^{(j)}-C^{\ominus}(\pi)^{(j)}-(b^d+f\tilde{\pi}b^f)\frac{\tilde{\omega}\ell^j}{(1-\tau_w)\lambda_G}]}{\{p(0)B\}C(\pi)^{(j)}}$$

(j=1, 2, ……, n) (57)

$$\tilde{r}_j=\frac{(1-\tau_r)[p(t)I^{(j)}-p(t-1)C^{\ominus}(\pi)^{(j)}-p(t-1)(b^d+f\tilde{\pi}b^f)\frac{\tilde{\omega}\ell^j}{(1-\tau_w)}]}{p(t-1)C(\pi)^{(j)}}$$

$$=\lim_{t\to\infty}\frac{(1-\tau_r)[\left\{p(t)/\lambda_G^t\right\}I^{(j)}-\left\{p(t-1)/\lambda_G^{t-1}\right\}\left\{C^{\ominus}(\pi)^{(j)}\frac{1}{\lambda_G}+(b^d+f\tilde{\pi}b^f)\frac{\tilde{\omega}\ell_j}{(1-\tau_w)\lambda_G}\right]}{\left\{p(t-1)/\lambda_G^{t-1}\right\}C(\pi)^{(j)}\frac{1}{\lambda_G}}$$

$$=\frac{(1-\tau_r)\{p(0)B\}[I^{(j)}-C^{\ominus}(\pi)^{(j)}\frac{1}{\lambda_G}+(b^d+f\tilde{\pi}b^f)\frac{\tilde{\omega}\ell_j}{(1-\tau_w)\lambda_G}]}{\{p(0)B\}C(\pi)^{(j)}\frac{1}{\lambda_G}}$$

(j=1, 2, ……, n) (58)

Comparing the right hand sides of the equations (57) and (58), we have $\lim_{t\to\infty}\tilde{r}_j^{**}(t)<\tilde{r}_j$ if $\lambda_G>1$.

(ii) Substituting eq. (41b)′ and eq. (51) into eq. (56), we have

$$\lim_{t\to\infty}\tilde{\omega}^{**}(t)=\lim_{t\to\infty}\frac{p(t-1)(b^d+f\tilde{\pi}b^f)}{p(t)(b^d+f\tilde{\pi}b^f)}\cdot\tilde{\omega}$$

$$=\lim_{t\to\infty}\frac{\left\{p(t-1)/\lambda_G^{t-1}\right\}(b^d+f\tilde{\pi}b^f)}{\left\{p(t)/\lambda_G^t\right\}(b^d+f\tilde{\pi}b^f)}\cdot\frac{\tilde{\omega}}{\lambda_G}$$

$$=\frac{\{p(0)B\}(b^d+f\tilde{\pi}b^f)}{\{p(0)B\}(b^d+f\tilde{\pi}b^f)}\cdot\frac{\tilde{\omega}}{\lambda_G}$$

$$=\frac{\tilde{\omega}}{\lambda_G}<\tilde{\omega}\qquad(\text{if }\lambda_G>1).$$ (59)

(q. e. d.)

Theorem 4.

(i) Suppose that one of the following conditions (a) ~ (c) is satisfied.

(a) An arbitrary $\tilde{r}_J$ is increased (decreased).

(b) An arbitrary element of the vector π or an element π_k of the vector $\tilde{\pi}$ such that $b^f_k > 0$ is increased (decreased).

(c) τ_r or τ_w is increased (decreased).

Then, other things being equal, λ_G increases (decreases) and $\lim_{t\to\infty} \tilde{\omega}_{**}(t)$ decreases (increases).

(ii) Suppose that $\tilde{\omega}$ is increased (decreased). Then, other things being equal, λ_G increases (decreases) and $\lim_{t\to\infty} \tilde{\omega}^{**}(t)$ increases (decreases).

(Proof.)

(i) This proposition directly follows from the Perron-Frobenius theorem and eq. (59).

(ii) Suppose that $\tilde{\omega}$ is increased. Then, λ_G must increase because of the Perron-Frobenius theorem. By the way, λ_G must satisfy the following 'characteristic equation'.

$$|\lambda_G I - G| \equiv |\lambda_G I - (H_1 + H_2\tilde{\omega})|$$

$$\equiv \lambda_G^n \left| I - \left(H_1 \frac{1}{\lambda_G} + H_2 \frac{\tilde{\omega}}{\lambda_G}\right) \right| = 0 \qquad (60)$$

where $H_1 \equiv C(\pi)\,[\hat{r}]\frac{1}{1-\tau_r} + C^{\ominus}(\pi) \geq 0$ and $H_2 \equiv (b^d + f\tilde{\pi}b^f)\,\ell\frac{1}{1-\tau_w} \geq 0$. Therefore, λ_G must vary so as to satisfy $\lambda_J = 1$, where λ_J is the Frobenius root of the indecomposable nonnegative matrix $J \equiv H_1\frac{1}{\lambda_G} + H_2\frac{\tilde{\omega}}{\lambda_G}$. Suppose that $(\tilde{\omega}/\lambda_G)$ is not increased when λ_G is increased. In this case, λ_J must decrease in view of the Perron-Frobenius theorem so that λ_J must become to be less than 1, which contradicts eq. (60). Therefore, $\tilde{\omega}/\lambda_G = \lim_{t\to\infty} \tilde{\omega}^{**}(t)$ must increase when $\tilde{\omega}$ is increased. The argument in the case where $\tilde{\omega}$ is decreased is quite symmetrical.

(q. e. d.)

It is somewhat difficult to obtain the clear results concerning the comparative dynamics for $\lim_{t\to\infty} \tilde{r}_j^{**}(t)$ in the general case, but, we can obtain the definite results under an additional special assumption.

Theorem 5.

Assume that the matrices $C(\pi)$ and $C^{\ominus}(\pi)$ satisfy the relationship $C^{\ominus}(\pi) = C(\pi)\,[\hat{\delta}]$, where $[\hat{\delta}] \equiv \begin{pmatrix} \delta_1 & & & 0 \\ & \delta_2 & & \\ & & \ddots & \\ 0 & & & \delta_n \end{pmatrix}$ (19). Then, we have the following propositions.

(i) Suppose that an arbitrary $\tilde{r}_j$ is increased (decreased).

Then, other things being equal, $\lim_{t\to\infty} r_j^{**}(t)$ increases (decreases) and all of $\lim_{t\to\infty} r_i^{**}(t)$ such that $i \neq j$ decreases (increases).

(ii) Suppose that either of the following conditions (a) or (b) is satisfied.

(a) An arbitrary element of the vector π or an element π_k of the vector $\tilde{\pi}$ such that $b_k^f > 0$ is increased (decreased).

(b) τ_r or τ_w is increased (decreased).

Then, other things being equal, $\lim_{t\to\infty} r_j^{**}(t)$ decreases (increases) for all j $\epsilon\{1, 2, \cdots\cdots, n\}$.

(Proof.)

(i) In this case, the equations (57) and (58) can be written as follows respectively.

$$\lim_{t\to\infty} \tilde{r}_j^{**}(t)=\frac{(1-\tau_r)\{p(0)B\}\left[I^{(j)}-(b^d+f\tilde{\pi}b^f)\frac{\tilde{\omega}\ell_j}{(1-\tau_w)\lambda_G}\right]}{\{p(0)B\}\ C(\pi)^{(j)}}$$
$$-(1-\tau_r)\delta_j \qquad (j=1,\ 2,\cdots\cdots,\ n) \qquad (57)'$$

$$\tilde{r}_j=\frac{(1-\tau_r)\ \{p(0)B\}\ \left[I^{(j)}-(b^d+f\tilde{\pi}b^f)\frac{\tilde{\omega}\ell_j}{(1-\tau_w)\lambda_G}\right]}{\{p(0)B\}C(\pi)^{(j)}\frac{1}{\lambda_G}}$$
$$-(1-\tau_r)\delta_j \qquad (j=1,\ 2,\ \cdots\cdots,\ n) \qquad (58)$$

From these equations we have

$$\lim_{t\to\infty} r_j^{**}(t)=(\tilde{r}_j/\lambda_G)+(1-\tau_r)\delta_j(1/\lambda_G-1) \quad (j=1,2,\cdots\cdots,n). \qquad (61)$$

Suppose that an arbitrary $\tilde{r}_j$ is increased (decreased). Then, λ_G increases (decreases) so that $\lim_{t\to\infty} \tilde{r}_i^{**}(t)$ such that $i \neq j$ must decrease (increase) in view of eq. (61). In this case, $\lim_{t\to\infty} \tilde{\omega}^{**}(t)$ also decreases (increases) from *Theorem 4* (i). Therefore, $\lim_{t\to\infty} \tilde{r}_j^{**}(t)$ must increase (decrease) in view of the characteristics of the after tax wage-profit surface.

(ii) It is obvious from the Perron-Frobenius theorem and eq. (61).

(q. e. d.)

Now, let us consider the aggregation of the disaggregated dynamical system (49). For this purpose, let us take up the right-Frobenius vector $x^{**}\equiv[x_1^{**},\ x_2^{**},\ \cdots\cdots,\ x_n^{**}]'>0$ of the indecomposable nonnegative matrix G. Namely,

$$[\lambda_G I-G]x^{**}=0\,. \tag{62}$$

The vector x^{**} may be considered to be the 'standard commodity' which is applied to the matrix G. Let us normalize x^{**} so as to satisfy the following condition.

$$\sum_{i=1}^{n} x_i^{**}=1 \tag{63}$$

Multiplying x^{**} from the right of eq. (49) and substituting eq. (62), we have

$$p(t)x^{**}=p(t-1)Gx^{**}=\lambda_G p(t-1)x^{**} \tag{64}$$

or equivalently,

$$\overline{p}_t=\lambda_G(r_1,\ \cdots\cdots,\ \tilde{r}_n,\ \tilde{\omega}\ ;\ \pi,\ \tilde{\pi},\ \tau_r,\ \tau_w)\overline{p}_{t-1} \tag{65}$$

where $\overline{p}_t\equiv p(t)\,x^{**}\equiv\sum_{i=1}^{n}p_i(t)\,x_i^{**}$ is the 'average' price level with the weight x^{**}. Note that the condition for the occurence of the wage-price inflation in this aggregated system is the same as that of the disaggregated system (49), i. e., $\lambda_G>1$. This fact assures that the simple one sector dynamic model (eq. (42)) is the correct 'surrogate' of the more complex disaggregated dynamic system (eq. (49)).

III-2. A Complication of the Model

In this section, we shall introduce a particular type of complication by considering the fact that the workers' response to the price change is apt to lag behind the capitalists' response. Now, let us replace eq. (41b)′ with the following equation.

$$w(t)=p(t-\theta)\ (b^d+f\tilde{\pi}b^f)\ \frac{\tilde{\omega}}{1-\tau_w} \tag{41b)″}$$

where θ is an integer such that $\theta \geqq 2$. Substituting eq. (41b)″ into eq. (41a)′, we have the following new dynamical system.

$$p(\mathrm{t})=p(\mathrm{t}-1)G_1(\hat{\hat{\mathrm{r}}}\ ;\ \pi,\ \tau_{\mathrm{r}})+p(\mathrm{t}-\theta)G_2(\tilde{\omega}\ ;\ \tilde{\mathrm{r}},\ \tau_{\mathrm{w}}) \tag{66}$$

where $G_1(\hat{\hat{\mathrm{r}}}\ :\ \pi,\ \tau_{\mathrm{r}}) \equiv C(\pi)\,[\hat{\hat{\mathrm{r}}}]\dfrac{1}{1-\tau_{\mathrm{r}}}+C^{\ominus}(\pi) \geq 0$ and $G_2(\tilde{\omega}\ ;\ \tilde{\pi},\ \tau_{\mathrm{w}}) \equiv (b^d+f\tilde{\pi}b^f)\,\ell\dfrac{\tilde{\omega}}{1-\tau_{\mathrm{w}}} \geq 0$. Needless to say, from the definition we have

$$G(\hat{\hat{\mathrm{r}}},\ \tilde{\omega}\ ;\ \pi,\ \tilde{\pi},\ \tau_{\mathrm{r}},\ \tau_{\mathrm{w}}) \equiv G_1(\hat{\hat{\mathrm{r}}}\ :\ \pi,\ \tau_{\mathrm{r}})+G_2(\tilde{\omega}\ ;\ \tilde{\pi},\ \tau_{\mathrm{w}}). \tag{67}$$

Now, we can transform eq. (66) into the following equivalent form by resorting to the usual procedure[20].

$$z(\mathrm{t})=z(\mathrm{t}-1)H(\hat{\mathrm{r}},\ \tilde{\omega}\ ;\ \pi,\ \tilde{\pi},\ \tau_{\mathrm{r}},\ \tau_{\mathrm{w}}) \tag{68}$$

where

$$z(\mathrm{t}) \equiv [p(\mathrm{t}),\ p(\mathrm{t}-1),\ \cdots\cdots,\ p(\mathrm{t}-\theta+1)] \tag{69}$$

and

$$H(\hat{\hat{\mathrm{r}}},\ \tilde{\omega}\ ;\ \pi,\ \tilde{\pi},\ \tau_{\mathrm{r}},\ \tau_{\mathrm{w}}) \equiv \underbrace{\begin{bmatrix} G_1(\hat{\hat{\mathrm{r}}}\ ;\ \pi,\ \tau_{\mathrm{r}}) & I & 0 & \cdots & 0 \\ 0 & 0 & I & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & & & \ddots & I \\ G_2(\tilde{\omega}\ ;\ \tilde{\pi},\ \tau_{\mathrm{w}}) & 0 & \cdots & \cdots & 0 \end{bmatrix}}_{\theta \mathrm{n}} \Bigg\} \theta \mathrm{n}. \tag{70}$$

Then, the characteristic equation of the dynamical system (68) is expresse as

$$\Delta(\lambda) \equiv \begin{vmatrix} \lambda I-G_1 & -I & 0 & \cdots & 0 \\ 0 & \lambda I & -I & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & \vdots & & \ddots & -I \\ -G_2 & 0 & \cdots & 0 & \lambda I \end{vmatrix} = 0. \tag{71}$$

The matrix H is a nonnegative matrix so that we can apply the Perron-Frobenius theorem. Namely, eq. (71) has at least one nonnegative eigenvalue and the largest nonnegative eigenvalue (the Frobenius root) λ_H satisfies the following condition.

$$\lambda_H \equiv \lambda_1 \geqq |\lambda_i| \quad \text{for all } i \in \{1,\ 2,\ \cdots\cdots,\ \theta n\} \tag{72}$$

where λ_is are the eigenvalues of eq. (71).

Now we can prove the following

Lemma 1.

(i) $\lambda_H > 0$ and there is no positive eigenvalue of eq. (71) other than λ_H.

(ii) $\lambda_H \gtreqless 1$ according as $\lambda_G \gtreqless 1$.

(Proof.)

(i) From eq. (71) and the characteristics of the determinant, we have the following relationship if $\lambda \neq 0$.

$$\Delta(\lambda) = \lambda^{\theta n} \begin{vmatrix} I - \frac{1}{\lambda} G_1 & -\frac{1}{\lambda} I & 0 & \cdots & 0 \\ 0 & I & -\frac{1}{\lambda} I & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & \vdots & \ddots & \ddots & -\frac{1}{\lambda} I \\ -\frac{1}{\lambda} G_2 & 0 & \cdots & 0 & I \end{vmatrix}$$

$$
=\lambda^{\theta n}\begin{vmatrix}
I-\frac{1}{\lambda}G_1 & -\frac{1}{\lambda}I & 0 & \cdots & \cdots & 0 \\
0 & I & -\frac{1}{\lambda}I & \ddots & & \vdots \\
\vdots & 0 & \ddots & \ddots & \ddots & 0 \\
0 & \vdots & \ddots & \ddots & -\frac{1}{\lambda}I & 0 \\
-\frac{1}{\lambda^2}G_2 & 0 & \cdots & 0 & I & 0 \\
-\frac{1}{\lambda}G_2 & 0 & \cdots & \cdots & 0 & I
\end{vmatrix}
$$

$$
= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
$$

$$
=\lambda^{\theta n}\begin{vmatrix}
I-\frac{1}{\lambda}G_1-\frac{1}{\lambda^{\theta}}G_2 & 0 & \cdots & \cdots & 0 \\
-\frac{1}{\lambda^{\theta-1}}G_2 & I & \ddots & & \vdots \\
\vdots & 0 & \ddots & \ddots & 0 \\
-\frac{1}{\lambda_2}G_2 & \vdots & \ddots & \ddots & \\
-\frac{1}{\lambda}G_2 & 0 & \cdots & 0 & I
\end{vmatrix}
$$

$$
=\lambda^{\theta n}\left| I-\frac{1}{\lambda}G_1 \quad -\frac{1}{\lambda^{\theta}}G_2 \right| \cdot \begin{vmatrix} I & & 0 \\ & \ddots & \\ 0 & & I \end{vmatrix}
$$

$$
=\lambda^{\theta n}\left| I-\left(\frac{1}{\lambda}G_1+\frac{1}{\lambda^{\theta}}G_2\right)\right| = 0 \qquad (73)
$$

Hence, the positive solution of eq. (71) is equivalent to the positive value of λ which assures that $\rho_k=1$, where ρ_k is the Frobenius root of the indecomposable nonnegative matrix $K(\lambda)\equiv\frac{1}{\lambda}G_1+\frac{1}{\lambda^{\theta}}G_2$[21]. It follows from the Perron-Frobenius theorem that ρ_k is the strictly decreasing continuous function of $\lambda > 0$. On the other hand, we have $\rho_K > 1$ for sufficiently small $\lambda > 0$ since the matrix $K(\lambda)$ is not productive for sufficiently small $\lambda > 0$ because of the fact $\lim_{\lambda \to 0} (1/\lambda) = \infty$, and we have $\rho_K < 1$ for sufficiently large $\lambda > 0$ because $\lim_{\lambda \to \infty} (1/\lambda) = 0$. Therefore, there exists the *unique* $\lambda > 0$ which assures that $\rho_K = 1$.

(ii) From the definitions of the characteristic equations, λ_G and λ_H must satisfy the following conditions.

$$\left| I - \left(\frac{1}{\lambda_G} G_1 + \frac{1}{\lambda_G} G_2\right) \right| = 0 \tag{74}$$

$$\left| I - \left(\frac{1}{\lambda_H} G_1 + \frac{1}{\lambda_H^\theta} G_2\right) \right| = 0 \tag{75}$$

Suppose that $\lambda_G > 1$. If $\lambda_H \leqq 1$, then, we have $\left(\frac{1}{\lambda_G} G_1 + \frac{1}{\lambda_G} G_2\right) \leq \left(\frac{1}{\lambda_H} G_1 + \frac{1}{\lambda_H^\theta} G_2\right)$, which implies that the Frobenius root of the matrix $\left(\frac{1}{\lambda_H} G_1 + \frac{1}{\lambda_H^\theta} G_2\right)$ is greater than that of the matrix $\left(\frac{1}{\lambda_G} G_1 + \frac{1}{\lambda_G} G_2\right)$. But, this is a contradiction because the equations (74) and (75) require that the value of either root must be one. This proves that $\lambda_G > 1 \Rightarrow \lambda_H > 1$. By using the similar reasoning, we can prove that $\lambda_G < 1 \Rightarrow \lambda_H < 1$ and $\lambda_G = 1 \Rightarrow \lambda_H = 1$.

(q. e. d.)

The following theorem is a simple economic interpretation of *Lemma 1*.

Theorem 6.

The system (66) can produce the wage-price inflation if and only if the combination $(\tilde{r}_1, \tilde{r}_2, \cdots\cdots, \tilde{r}_n, \tilde{\omega})$ lies above the after tax wage-profit surface. In other words, the condition for the occurence of the wage-price inflation in the system (66) is the same as that in the system (49).

Now, let us consider the case of $\lambda_H > 1$. Although the matrix G_1 is indecomposable, the matrix H is not necessarily indecomposable[22]. Therefore, we can not exclude the possibility that the relative prices continue to oscillate without converging to some limits. This phenomenon can occur if there exists $i \neq 1$ such that $\lambda_H \equiv \lambda_1 = |\lambda_i|$. However, for simplicity's sake, we shall *assume* a priori as follows.

Assumption 7. $\lambda_H \equiv \lambda_1 > |\lambda_i|$ if $i \neq 1$.

In this case, λ_H becomes to be the 'dominant root' so that the rate of inflation of the price of each good will approach to $\lambda_H - 1 > 0$ ultimately if the initial condition $z(0)$ is appropriate.

Now, the after tax real wage rate $\tilde{\omega}^{**}(t)$ is expressed as follows in view of eq. (41b)''.

$$\begin{aligned}\tilde{\omega}^{**}(t) &\equiv \frac{(1-\tau_w)\,w(t)}{p(t)(b^d + f\tilde{\pi}b^f)} \\ &= \frac{p(t-\theta)(b^d + f\tilde{\pi}b^f)}{p(t)(b^d + f\tilde{\pi}b^f)}\tilde{\omega}\end{aligned} \tag{76}$$

Hence, under the above assumption, we have

$$\lim_{t\to\infty} \tilde{\omega}^{**}(t) = \frac{\left\{p(t)/\lambda_H^\theta\right\}(b^d + f\bar{\pi}b^f)}{p(t)(b^d + f\bar{\pi}b^f)}\tilde{\omega} = \frac{\tilde{\omega}}{\lambda_H^\theta} \tag{77}$$

Lemma 2.

Suppose $\lambda_G > 1$ so that $\lambda_H > 1$. Then, other things being equal, (i) the larger θ, the smaller is λ_H, and (ii) the larger θ, the larger is λ_H^θ.

(Proof.)

(i) Suppose that $\theta_1 < \theta_2$ and $1 < \lambda_{H1} \leqq \lambda_{H2}$, where λ_{Hi} is the Frobenius root of the matrix H which is accompanied by θ_i. Then, we have $(\frac{1}{\lambda_{H1}}G_1 + \frac{1}{\lambda_{H1}{}^{\theta_1}}G_2) \geq (\frac{1}{\lambda_{H2}}G_1 + \frac{1}{\lambda_{H2}{}^{\theta_2}}G_2)$ so that the Frobenius root of the matrix $(\frac{1}{\lambda_{H1}}G_1 + \frac{1}{\lambda_{H1}{}^{\theta_1}}G_2)$ must be greater than that of the matrix $(\frac{1}{\lambda_{H2}}G_1 + \frac{1}{\lambda_{H2}{}^{\theta_2}}G_2)$. But, this is a contradiction because both matrix must have the common Frobenius root $\rho = 1$ in view of eq. (75). Therefore, we have $\lambda_{H1} > \lambda_{H2} > 1$.

(ii) Suppose that $\theta_1 < \theta_2$. Then, we have $\lambda_{H1} > \lambda_{H2} > 1$ from (i). Furthermore, suppose that $\lambda_{H1}{}^{\theta_1} \geqq \lambda_{H2}{}^{\theta_2}$. In this case, we have $(\frac{1}{\lambda_{H1}}G_1 + \frac{1}{\lambda_{H1}{}^{\theta_1}}G_2) \leq (\frac{1}{\lambda_{H2}}G_1 + \frac{1}{\lambda_{H2}{}^{\theta_2}}G_2)$, which implies that the Frobenius root of $(\frac{1}{\lambda_{H1}}G_1 + \frac{1}{\lambda_{H1}{}^{\theta_1}}G_2)$ is smaller than that of $(\frac{1}{\lambda_{H2}}G_1 + \frac{1}{\lambda_{H2}{}^{\theta_2}}G_2)$. But, this contradicts eq. (75). Hence, we must have $\lambda_{H1}{}^{\theta_1} < \lambda_{H2}{}^{\theta_2}$.

(q. e. d.)

Theorem 7.

Suppose that the combination $(\tilde{r}_1, \tilde{r}_2, \cdots\cdots, \tilde{r}_n, \tilde{\omega})$ is above the after tax wage-profit surface and *Assumption 7* is satisfied. Then, if the time lag of the workers' response is increased (decreased), the ultimate rate of inflation decreases (increases) and $\lim_{t\to\infty} \tilde{\omega}^{**}(t)$ decreases (increases).

(Proof.)

These propositions directly follow from eq. (77) and *Lemma 2.*

(q. e. d.)

It is easily seen, in passing, that the theorems 4 and 5 in the previous section are still effective in the present model under *Assumption 7*.

Lastly, let us consider the aggregation of this system. The right-Frobenius vector $\bar{x}\equiv[\bar{x}_1, \bar{x}_2, \cdots\cdots, \bar{x}_{\theta n}]'\geq 0$ of the matrix $H\geq 0$ can be written as

$$[\lambda_H I-H]\bar{x}=0\,. \tag{78}$$

We shall also normalize $\bar{x}$ so as to satisfy the condition

$$\sum_{i=1}^{\theta n}\bar{x}_i=1\,. \tag{79}$$

Substituting eq. (78) into eq. (69), we obtain

$$\bar{z}_t=\lambda_H(\hat{r},\ \tilde{\omega}\ ;\ \pi,\ \tilde{\pi},\ \tau_r,\ \tau_W)\ \bar{z}_{t-1} \tag{80}$$

where $\bar{z}_t\equiv z(t)\,\bar{x}$. $\bar{z}_t$ is a weighted average of the prices in the period $(t-\theta+1)$ through the period t. Eq. (80) shows that eq. (42) in the simple one sector model still serves as a 'surrogate' of rather complicated system in this section.

IV. A Dual Analysis ; Many Goods Multiplier Process and the 'Standard Price'

Up to the previous section, we have investigated the disaggregated dynamics of the price system and its aggregation in terms of the standard commodity. In this section, we shall consider the 'dual' of the previous analysis, i. e., the disaggregated dynamics and the aggregation of the quantity system. For this purpose, let us take up the multisectoral version of the Keynesian multiplier process which was formulated by Morishima (1976) chap. 8.

Morishima's so called 'Leontief-Keynes process' is formulated as follows.

$$x(t)=A^{\ominus}x(t-1)+D(t-1)+g \qquad (81)$$

where $A^{\ominus}\geq 0$ is the indecomposable capital depreciation matrix, $x(t)\equiv[x_1(t),\ x_2(t),\ \cdots\cdots,\ x_n(t)]'$ is the commodity output vector, $D(t)\equiv[D_1(t),\ D_2(t),\ \cdots\cdots,\ D_n(t)]'$ is the effective consumption demand vector and $g\equiv[g_1,\ g_2,\ \cdots\cdots,\ g_n]'\geq 0$ is the vector of the 'autonomous demands' which include the firms' investment expenditures for the fixed capitals and the government expenditures(23). Following Morishima (1976) and the Keynesian tradition, let us assume that $D_i(t)$ is a simple linear function of the real national income, i. e.,

$$D_i(t)=c_i(\bar{p}(t))(1-\tau)\bar{Y}(t)+d_i$$
$$\equiv c_i(\bar{p}(t))(1-\tau)\bar{p}(t)[I-A^{\ominus}]x(t)+d_i \qquad (82)$$
$$(i=1,\ 2,\ \cdots\cdots,\ n)$$

where $\bar{p}(t)\equiv p(t)/w(t)=[p_1(t)/w(t),\ p_2(t)/w(t),\ \cdots\cdots,\ p_n(t)/w(t)]$ is the price vector and $\bar{Y}(t)=\bar{p}(t)[I-A^{\ominus}]x(t)$ is the pre tax net national income, both are measured in terms of the 'wage unit' following Keynes' (1936) suggestion. τ is the average income tax rate $(0\leq\tau<1)$.

Substituting eq. (82) into eq. (81) gives

$$x(t)=[A^{\ominus}+(1-\tau)c(\bar{p}(t-1))\bar{p}(t-1)[I-A^{\ominus}]]x(t-1) + (d+g) \tag{83}$$

where $c(\bar{p}(t))\equiv[c_1[\bar{p}(t), c_2(\bar{p}(t), \cdots\cdots, c_n(\bar{p}(t))]'\geq 0$ is the vector of the consumption coefficients and $d\equiv[d_1, d_2, \cdots\cdots, d_n]'\geqq 0$ is the 'autonomous' consumption vector.

Now, let us assume that $\bar{p}(t)$ is positive and constant through time and $\bar{p}(t)>\bar{p}(t)A^{\ominus}$, namely,

$$\bar{p}(t)=\bar{p}>0 \tag{84}$$

and

$$\bar{p}[I-A^{\ominus}]>0\,. \tag{85}$$

These conditions are consistent with the steady state of the wage-price inflation analyzed in the previous sections. Substituting these conditions into eq. (83), we have

$$x(t)=Vx(t-1)+h \tag{86}$$

where $V\equiv[A^{\ominus}+(1-\tau)c(\bar{p})\bar{p}[I-A^{\ominus}]]\geq 0$ and $h\equiv d+g\geq 0$.

We can easily find the solution of eq. (86) by the simple iteration, i. e.,

$$x(t)=[I+V+V^2+\cdots\cdots+V^{t-1}]h+V^t x(0) \tag{87}$$

It is well known that this process is stable if and only if the nonnegative matrix V is 'productive' so that the Frobenius root λ_v is less than one.

Morishima (1976) proved that this condition is satisfied under the standard Keynesian assumption that the 'marginal propensity to consume' is less than one as follows.

From eq. (82) we have

$$\bar{p}D(t)=\bar{p}c(\bar{p})(1-\tau)\bar{Y}(t)+\bar{p}d \tag{88}$$

so that

$$(1-\tau)\bar{p}c(\bar{p})=\Delta(\bar{p}D(t))/\Delta\bar{Y}(t) \tag{89}$$

which implies that $(1-\tau)\bar{p}c(\bar{p})$ is the marginal propensity to consume. Now, let us assume that

$$(1-\tau)\bar{p}c(\bar{p})<1. \tag{90}$$

In this case, we have

$$\begin{aligned}\bar{p}V&=\bar{p}A^{\ominus}+(1-\tau)\bar{p}c(\bar{p})\bar{p}[I-A^{\ominus}]\\&<\bar{p}A^{\ominus}+\bar{p}[I-A^{\ominus}]=\bar{p}\end{aligned} \tag{91}$$

where $\bar{p}>0$ by assumption. Eq. (91) implies that the matrix $V\geq 0$ is productive, so that the adjustment process (86) is stable. In this case, we have $\lim_{t\to\infty} V^t=0$ and $[I+V+V^2+\cdots\cdots]=[I-V]^{-1}>0$ (24), so that from eq. (87) we can conclude that

$$\lim_{t\to\infty} x(t)=[I+V+V^2+\cdots\cdots]h=[I-V]^{-1}h=\bar{x}>0. \tag{92}$$

$[I-V]^{-1}$ is nothing but the so called 'matrix multiplier', because we can see the effects of the variations of the autonomous expenditures by using the following simple formula.

$$\Delta\bar{x}=[I-V]^{-1}\Delta h \tag{93}$$

Now, we shall consider the aggregation of the system (86). For this purpose, let us consider the left-Frobenius vector $p^*\equiv[p_1^*, p_2^*, \cdots\cdots, p_n^*]>0$ of the matrix V, namely,

$$p^*[\lambda_v I-V]=0 \quad ; \quad \sum_{i=1}^{n} p_i^*=1. \tag{94}$$

From eq. (86) and eq. (94) we have

$$p^*x(t)=p^*Vx(t-1)+p^*h=\lambda_v p^*x(t-1)+p^*h \tag{95}$$

or equivalently,

$$y_t=\lambda_v y_{t-1}+h^* \tag{96}$$

where $y_t\equiv p^*x(t)$, $h^*\equiv p^*h$ and

$$0<\lambda_v=\lambda_v(\underset{\ominus}{\tau},\ c(\underset{\oplus}{\bar{p}}))<1. \tag{97}$$

The aggregator p* is the 'dual' of the Sraffian standard commodity which is the right-Frobenius vector of the relevant matrix, so that we shall call p^* the 'standard price' vector which is distinct from the 'actual' price vector $\bar{p}$. y_t is the gross national product which is measured in terms of the 'standard prices'. Needless to say, eq. (96) represents the adjustment process of the conventional one sector Keynesian model, and λ_v is the marginal propensity to consume in this aggregated system. Note that the stability condition of the system (96) ($\lambda_v < 1$) coincides with that of the disaggregated counterpart (eq. (86)).

Eq. (96) can be solved by the usual way, namely,

$$y_t = (1+\lambda_v+\lambda_v^2+\cdots\cdots+\lambda_v^{t-1})h^* + y_0\lambda_v^t$$

$$= \frac{1-\lambda_v^t}{1-\lambda_v}h^* + y_0\lambda_v^t \tag{98}$$

and

$$\lim_{t\to\infty} y_t = (1+\lambda_v+\lambda_v{}^2+\cdots\cdots)\,h^* = \frac{1}{1-\lambda_v}h^* \equiv \bar{y} > 0\,. \tag{99}$$

This process can be illustrated by the graphical device which is popular in the elementary textbooks of Macroeconomics (see Fig. 6).

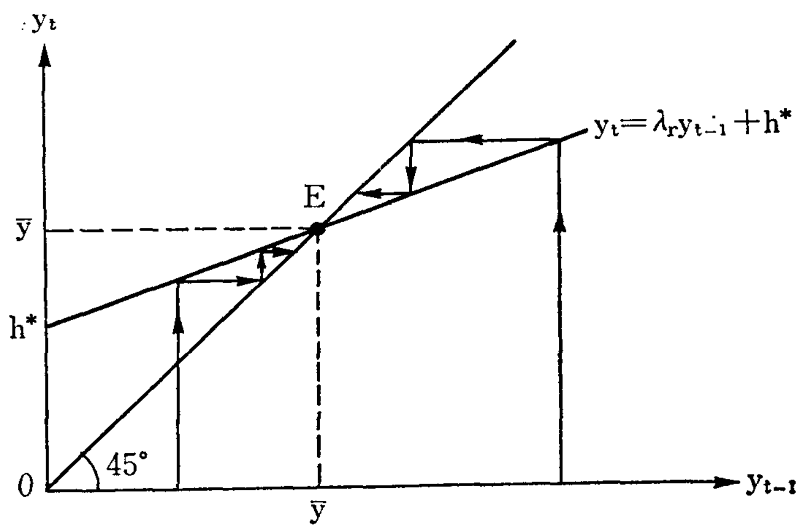

Fig. 6.

By the way, from eq. (99) we obtain the usual (scalar) multiplier formula, i. e.,

$$\frac{\Delta \bar{\mathrm{y}}}{\Delta \mathrm{h}^*}\equiv\frac{p^*(\Delta \bar{x})}{p^*(\Delta h)}=\frac{1}{1-\lambda_{\mathrm{v}}}>1\,. \tag{100}$$

Hence, in a sense, the simple one sector Keynesian model in the elementary textbooks is a correct 'surrogate' of the multisectoral Keynesian system.

The above analysis neglected the difference of the consumption patterns from the wage income and the profit income following the textbook Keynesian model. However, we can show that the basic conclusion of the previous analysis is not affected even if we explicitly consider the difference of the consumption patterns from two income categories following Post Keynesian or Marxian tradition.

Now, let us modify eq. (81) and eq. (82) as follows respectively.

$$x(\mathrm{t})=A^{\ominus}x(\mathrm{t}-1)+D^{\mathrm{w}}(\mathrm{t}-1)+D^{\mathrm{r}}(\mathrm{t}-1)+g \tag{81$'$}$$

$$\mathrm{D}_{\mathrm{i}}^{\mathrm{w}}(\mathrm{t})=\mathrm{c}_{\mathrm{i}}^{\mathrm{w}}(\bar{\mathrm{p}}(\mathrm{t}))(1-\tau_{\mathrm{W}})\,\bar{\mathrm{Y}}_{\mathrm{W}}(\mathrm{t})+\mathrm{d}_{\mathrm{i}}^{\mathrm{W}} \tag{82$'$}$$

$$\mathrm{D}_{\mathrm{i}}^{\mathrm{r}}(\mathrm{t})=\mathrm{c}_{\mathrm{i}}^{\mathrm{r}}(\bar{\mathrm{p}}(\mathrm{t}))(1-\tau_{\mathrm{r}})\,\bar{\mathrm{Y}}_{\mathrm{r}}(\mathrm{t})+\mathrm{d}_{\mathrm{i}}^{\mathrm{r}}$$

$$(\mathrm{i}=1,\ 2,\ \cdots\cdots,\ \mathrm{n})$$

where $D^{\mathrm{w}}(\mathrm{t})=[\mathrm{D}_1^{\mathrm{w}}(\mathrm{t}),\ \mathrm{D}_2^{\mathrm{w}}(\mathrm{t}),\ \cdots\cdots,\ \mathrm{D}_{\mathrm{n}}^{\mathrm{w}}(\mathrm{t})]'$ and $D^{\mathrm{r}}(\mathrm{t})\equiv[\mathrm{D}_1^{\mathrm{r}}(\mathrm{t}),\ \mathrm{D}_2^{\mathrm{r}}(\mathrm{t}),\ \cdots\cdots,\ \mathrm{D}_{\mathrm{n}}^{\mathrm{r}}(\mathrm{t})]'$ are the effective consumption demand vectors from the wage income and the profit income. Furthermore, $\bar{\mathrm{Y}}_{\mathrm{w}}(\mathrm{t})$ and $\bar{\mathrm{Y}}_{\mathrm{r}}(\mathrm{t})$ are the pre tax wage income and the pre tax profit income which are measured in terms of the 'wage unit'.

Next, let us assume the following price system.

$$p(\mathrm{t})=p(\mathrm{t})A[\hat{\mathrm{r}}]+p(\mathrm{t})A^{\ominus}+\mathrm{w}(\mathrm{t})\ell \tag{101}$$

or,

$$\bar{p}=\bar{p}A[\hat{\mathrm{r}}]+\bar{p}A^{\ominus}+\ell \tag{101$'$}$$

where $\bar{p}>0$, $\bar{p}A[\hat{\mathrm{r}}]>0$, $\ell>0$ and $[\hat{\mathrm{r}}]$ is constant through time.

In this case, $\bar{p}$ also becomes to be constant. Obviously, this situation is consistent with the steady state of wage-price inflation. In this case, we have

$$\bar{Y}_w(t)=\ell x(t) \tag{102}$$

and

$$\bar{Y}_r(t)=\bar{p}A[\hat{r}]x(t). \tag{103}$$

Substituting the equations (82)′, (101)′, (102) and (103) into eq. (81)′, we obtain the following system.

$$\begin{aligned} x(t)&=[A^{\ominus}+(1-\tau_w)c^w(\bar{p})\ell+(1-\tau_r)c^r(\bar{p})\bar{p}A[\hat{r}]]x(t-1)\\ &\quad+(d^w+d^r+g)\\ &\equiv\tilde{V}x(t-1)+\tilde{h} \end{aligned} \tag{104}$$

where $\tilde{V}\equiv[A^{\ominus}+(1-\tau_w)c^w(\bar{p})\ell+(1-\tau_r)c^r(\bar{p})\,\bar{p}A[\hat{r}]]\geq 0$, $c^w(\bar{p})\equiv[c_1^w(\bar{p}), c_2^w(\bar{p}), \cdots\cdots, c_n^w(\bar{p})]'\geq 0$, $c^r(\bar{p})\equiv[c_1^r(\bar{p}), c_2^r(\bar{p}), \cdots\cdots, c_n^r(\bar{p})]'\geq 0$, $d^w\equiv[d_1^w, d_2^w, \cdots\cdots, d_n^w]'\geqq 0$, $d^r\equiv[d_1^r, d_2^r, \cdots\cdots, d_n^r]'\geqq 0$ and $\tilde{h}\equiv d^w+d^r+g\geq 0$.

We can easily confirm that $(1-\tau_w)\bar{p}\,c^w(\bar{p})$ and $(1-\tau_r)\bar{p}\,c^r(\bar{p})$ are the marginal propensities to consume from the wage income and the profit income respectively. Now, we shall assume that

Assumption 8. $(1-\tau_w)\bar{p}c^w(\bar{p})\leqq 1$ and $(1-\tau_r)\bar{p}c^r(\bar{p})<1$.

Theorem 8.

Under *Assumption 8*, the matrix $\tilde{V}$ is productive so that the system (104) is stable.

(Proof.)

From the definition of $\tilde{V}$ and eq. (101)′, we have

$$\bar{p}\tilde{V}=\bar{p}A^{\ominus}+(1-\tau_{w})\bar{p}c^{w}(\bar{p})\ell+(1-\tau_{r})\bar{p}c^{r}(\bar{p})\bar{p}A[\hat{r}]$$

$$<\bar{p}A^{\ominus}+\ell+\bar{p}A[\hat{r}]=\bar{p}\ ;\ \bar{p}>0, \tag{105}$$

which implies that $\tilde{V}$ is productive.

(q. e. d.)

Now, let us consider the left-Frobenius vector $p^{**}>0$ of $\tilde{V}$, i. e.,

$$p^{**}[\lambda_{\tilde{V}}I-\tilde{V}]=0 \tag{106}$$

where

$$0<\lambda_{\tilde{V}}\equiv\lambda_{\tilde{V}}(\underset{\ominus}{\tau_{w}},\ \underset{\ominus}{\tau_{r}},\ \underset{\oplus}{c^{w}}(\bar{p}),\ \underset{\oplus}{c^{r}}(\bar{p}))<1. \tag{107}$$

From the equations (104) and (106), we obtain the aggregated system

$$\tilde{y}_{t}=\lambda_{\tilde{V}}\tilde{y}_{t-1}+h^{**} \tag{108}$$

where $\tilde{y}_{t}\equiv p^{**}x(t)$ and $h^{**}\equiv p^{**}\tilde{h}$. Qualitalively eq. (108) is the same as eq. (96) so that we need not repeat the further analysis.

V. Concluding Remarks

In this paper we have presented some examples of the applications of the Sraffian idea to the economic dynamics. It must be noted, incidentally, that we investigated the dynamics of the price system and the quantity system separately. Without doubt, this is not a satisfactory way to analyze the working of the economy as a whole. But, there is some reality in the following assertion by Goodwin.

"The fact that output is kept constant in the analysis of price levels and that price is kept constant in dealing with output is not as inconsistent as it seems. It is merely a device for reducing a non-linear problem to the more tractable form of a pair of linear ones. Nearly simultaneous variations of the twin motions can be studied. First price is held constant and then output is held constant to find the change in price. The process is then repeated, successively. In this manner the parallel but somewhat independent behaviour of both is obtained". (Goodwin (1983) p. 51)

For example, we can study the effects of the change of the tax policy by the government as follows in line with Goodwin's suggestion.

Suppose that the government enforces the tax increase, i. e., τ_r or τ_w is increased. According to the model of the price system presented in section III, the direct effect of this policy is the acceleration of the inflation. However, this policy also affects the state of the effective demand. The model of the quantity system which was presented in section IV shows that the tax increase induces the reduction of the effective demand through the reduction of the value of each element of the matrix multiplier, so that the level of employment must decrease. This fact may have some feedback effects on the price system. First, the increase of the unemployment may weaken the bargaining power of the workers so the parameter $\bar{\omega}$ may decrease. Furthermore, the decrease of the national income may induce the improvement of the balance of payment through the reduction of the import. This fact may contribute to the improvement of the terms of trade through the rise of the value of the domestic currency relative to the foreign currency under the floating exchange rate system. These indirect feedback effects through the quantity system has the depressing rather than the accelerating effect on the wage-price inflation process. Therefore, the tax increase may or may not accelerate the inflation. Furthermore, the changes of the relative prices in this process will have some feedback effects on the quantity system[25].

If we formulate this system formally, we will have a system of the nonlinear difference equations with many variables, because we must introduce some sorts of nonlinearity into the system to consider the interdependence of the price system and the quantity system explicitly. Behavior of such a system may be very complicated. For example, even the 'chaotic' motion can emere[26]. It is beyond the scope of this paper to trace the behavior of such a system in detail. This is the theme which is left to the future investigation. But, Sraffa's method of the aggregation may contribute to simplify the analysis of such a system considerably[27].

<Appendix ; Labor Value and Morishima-Seton Equation in an Open Economy>

In this appendix we shall concentrate on a special topic which was not considered in the text, i. e., the Marxian theory of value and exploitation in an open economy.

A-1. Labor Value and the Definition of the Rate of Exploitation

We shall retain the notation and the assumptions which are adopted in section III of the text. Then, the system of Marxian 'labor value' equations in our open economy may be formulated as follows.

$$\Lambda=\Lambda A\Theta+\Lambda f\pi M\Theta+\ell \qquad \text{(A1)}$$

where $\Lambda \equiv [\Lambda_1, \Lambda_2, \cdots\cdots, \Lambda_n]$ is the vector of the labor values of the domestically produced commodities. The first part of the right hand side of eq. (A1) shows that the quantity of the (direct and indirect) domestic labor for the replacement of the domestically produced capital goods, while the second part is the quantity of the (direct and indirect) domestic labor for the production of the export goods which can be exchanged with the imported capital goods for replacement in the international market.

Obviously, eq. (A1) is a natural extension of the notion of the labor values in the closed system into the open system(28). From *Assumption* 5 in the text we have $[I-(A\Theta+f\pi M\Theta)]^{-1} \geq 0$, so that eq. (A1) can be solved as follows(29).

$$\Lambda = \ell[I-(A\Theta+f\pi M\Theta)]^{-1} = \ell\sum_{t=0}^{\infty}(A\Theta+f\pi M\Theta)^t > 0 \tag{A2}$$

Then the rate of exploitation (e) is naturally defined as

$$e \equiv \frac{1-\omega(\Lambda b^d+\Lambda f\bar{\pi}b^f)}{\omega(\Lambda b^d+\Lambda f\bar{\pi}b^f)} > -1. \tag{A3}$$

Now, we can consider an alternative definition of the rate of exploitation (e′) which is in line with Morishima (1974)'s approach, namely,

$$e' \equiv \frac{N-\ell x^0}{\ell x^0} \tag{A4}$$

where $N > 0$ is the 'actual' labor time and x^0 is an optimal solution of the following linear programming problem.

$$\text{Minimize } \ell x \text{ subject to } x \geq A\Theta x + f\pi M\Theta x + \omega(b^d+f\bar{\pi}b^f)N, \quad x \geq 0 \tag{A5}$$

But, it is easy to show that $\ell x^0 = \omega\Lambda(b^d+f\bar{\pi}b^f)N$ so that $e = e'$(30). That is, the alternative definitions of the rate of exploitation we have considered above are identical each other.

A-2. Morishima-Seton Equation

It is well known that Morishima and Seton (1961) derived the Marxian equality

$$r = \frac{e\,V^*}{C^*+V^*} \tag{A6}$$

in a framework of the circulating capital model with equal rate of profit in the closed economy, where r is the pre tax rate of profit, and C* and V* are weighted averages of the so called 'constant capitals' and 'variable capitals' which are caluculated in terms of the labor values respectively. Now, let us reconsider the Morishima-Seton equation in our analytical framework.

Marxian 'prices of production' system with equal rate of profit in an open economy can be expressed as(31)(32).

$$p = \mathrm{r}\, p\{A+f\pi M+\omega(b^d+f\bar{\pi}b^f)\ell\}+p\{A\ominus+f\pi M\ominus+\omega(b^d+f\bar{\pi}b^f)\ell\} \quad \text{(A7)}$$

or equivalently,

$$p[I-\mathrm{r}\{A+f\pi M+\omega(b^d+f\bar{\pi}b^f)\ell\}-\{A\ominus+f\pi M\ominus+\omega(b^d+f\bar{\pi}b^f)\ell\}] = 0. \quad \text{(A7)}'$$

Eq. (A7)$'$ implies that the vector of the 'prices of production' $p>0$ can be expressed as the left-Frobenius vector of the matrix $Q\equiv \mathrm{r}\{A+f\pi M+\omega(b^d+f\bar{\pi}b^f)\}+\{A\ominus+f\bar{\pi}M\ominus+\omega(b^d+f\bar{\pi}b^f)\}$(33). Then, we can consider the 'dual' of p, i. e., the right-Frobenius vector $x^*>0$ of the matrix Q.

$$[I-\mathrm{r}\{A+f\pi M+\omega(b^d+f\bar{\pi}b^f)\ell\}-\{A\ominus+f\pi M\ominus+\omega(b^d+f\bar{\pi}b^f)\ell]x^* = 0 \quad \text{(A8)}$$

It is easily seen that x^* represents the output composition of the 'von Neumann path' with the equal rate of growth r.

Next, from eq. (A3) we have

$$(1+\mathrm{e})\Lambda\omega(b^d+f\bar{\pi}b^f) = 1. \quad \text{(A9)}$$

Substituting eq. (A9) into eq. (A1), we obtain

$$\Lambda[I-\{A\ominus+f\pi M\ominus+(1+\mathrm{e})\omega(b^d+f\bar{\pi}b^f)\ell\}] = 0. \quad \text{(A10)}$$

This equation implies that the labor value vector $\Lambda>0$ can be expressed as the left-Frobenius vector of the matrix $T\equiv A\ominus+f\pi M\ominus+(1+\mathrm{e})\omega(b^d+f\bar{\pi}b^f)\ell\geq 0$(34). Then, we can consider the 'dual' of Λ, i. e., the right-Frobenius vector $x^{**}>0$ of the matrix T.

$$[I-\{A\ominus+f\pi M\ominus+(1+\mathrm{e})\omega(b^d+f\bar{\pi}b^f)\ell\}]x^{**} = 0 \quad \text{(A11)}$$

Theorem A1.

$$\mathrm{r} = \frac{\mathrm{e}V_v{}^*}{C_v{}^*+V_v{}^*} = \frac{\mathrm{e}V_p{}^*}{C_p{}^*+V_p{}^*} \quad \text{(A12)}$$

where $C_v^* \equiv \Lambda(A+f\pi M)x^*$, $V_v^* \equiv \Lambda\omega(b^d+f\bar{\pi}b^f)\ell x^*$,
$C_p^* \equiv p(A+f\pi M)x^{**}$, and $V_p^* \equiv p\omega(b^d+f\bar{\pi}b^f)\ell x^{**}$.

(Proof.)

(i) Pre-multiplying eq. A8 by $\Lambda > 0$ and rearranging terms, we have

$$\Lambda[I-\{A\ominus+f\pi M\ominus+\omega(b^d+f\bar{\pi}b^f)\ell\}x^*$$
$$= r\,\Lambda\{A+f\pi M+\omega(d^d+f\bar{\pi}b^f)\ell\}x^* \equiv r(C_v^*+V_v^*). \qquad (A13)$$

On the other hand, post-multiplying eq. (A10) by $x^* > 0$ and rearranging terms, we have

$$\Lambda[I-\{A\ominus+f\pi M\ominus+\omega(b^d+f\bar{\pi}b^f)\ell\}x^*$$
$$= e\Lambda\omega(b^d+f\bar{\pi}b^f)\ell x^* \equiv eV_v^*. \qquad (A14)$$

Comparing the equations (A13) and (A14), we have
$r = eV_v^*/(C_v^*+V_v^*)$.

(ii) Post-multiplying eq. (A7)′ by $x^{**} > 0$ and rearranging terms, we have

$$p[I-\{A\ominus+f\pi M\ominus+\omega(b^d+f\bar{\pi}b^f)\ell\}]x^{**}$$
$$= rp\{A+f\pi M+\omega(b^d+f\bar{\pi}b^f)\ell\}x^{**}$$
$$\equiv r\,(C_p^*+V_p^*). \qquad (A15)$$

On the other hand, pre-multiplying eq. (A11) by $p > 0$ and rearranging, we have

$$p[I-\{A\ominus+f\pi M\ominus+\omega(b^d+f\bar{\pi}b^f)\ell\}x^{**}$$
$$= ep\omega(b^d+f\bar{\pi}b^f)\ell x^{**} \equiv eV_p^*. \qquad (A16)$$

Comparing eq. (A15) and (A16), we have

$r = eV_p^*/(C_p^*+V_p^*)$.

(q. e. d.)

Corollary A1.

$r > 0$ if and only if $e > 0$.

The first equality of eq. (A12) is nothing but the Morishima-Seton equation in our model. Corollary of *Theorem A1* is nothing but so called 'Fundamental Marxian theorem' in an open economy.

Finally, we shall consider the case with differential rates of profit. Any price system with differential rates of profit may be written as follows.

$$p[I-\{A+f\pi M+\omega(b^d+f\tilde{\pi}b^f)\ell\}[\hat{\mathrm{r}}]-\{A\Theta+f\pi M\Theta+\omega(b^d+f\tilde{\pi}b^f)\ell\}=0 \tag{A17}$$

where

$$[\hat{\mathrm{r}}]=\begin{pmatrix} \mathrm{r}_1 & & 0 \\ & \mathrm{r}_2 & \\ & & \ddots \\ 0 & & \mathrm{r}_n \end{pmatrix} \tag{A18}$$

is the diagonal matrix of the pre tax rates of profit (we need not assume that r_is are nonnegative although we assume that $p>0$). Then, we can define the 'average pre tax rate of profit' $\bar{\mathrm{r}}$ by using the vector x^{**} as follows.

$$\begin{aligned}\bar{\mathrm{r}} &\equiv \frac{px^{**}-p(A\Theta+f\pi M\Theta)x^{**}-p\omega(b^d+f\tilde{\pi}b^f)\ell x^{**}}{p(A+f\tilde{\pi}M)x^{**}+p\omega(b^d+f\pi b^f)\ell x^{**}} \\ &\equiv \frac{p\{A+f\pi M+\omega(b^d+f\tilde{\pi}b^f)\ell\}[\hat{\mathrm{r}}]x^{**}}{p\{A+f\pi M+\omega(b^d+f\pi b^f)\ell\}x^{**}}\end{aligned} \tag{A19}$$

Theorem A2.

$$\bar{\mathrm{r}}=\frac{\mathrm{eV_p}^*}{\mathrm{C_p}^*+\mathrm{V_p}^*} \tag{A20}$$

where $\mathrm{C_p}^*\equiv p(A+f\pi M)x^{**}$ and $\mathrm{V_p}^*\equiv p\omega(b^d+f\tilde{\pi}b^f)\ell x^{**}$.

(Proof.)

Pre-multiplying eq. (A11) by $p>0$ and rearranging terms, we have

$$\begin{aligned}&p[I-\{A\Theta+f\pi M\Theta+\omega(b^d+f\tilde{\pi}b^f)\ell\}]x^{**} \\ &=\mathrm{e}p\omega(b^d+f\tilde{\pi}b^f)\ell x^{**}\equiv \mathrm{eV_p}^*.\end{aligned} \tag{A21}$$

Dividing both sides of this equation by $p\{A+f\pi M+\omega(b^d+f\tilde{\pi}b^f)\ell\}x^{**}\equiv\mathrm{C_p}^*+\mathrm{V_p}^*$ and considering eq. (A19), we obtain eq. (A20).

(q. e. d.)

Corollary A2.

$\bar{r}>0$ if and only if $e>0$.

Therefore, the Fundamental Marxian theorem can be extended to the open economy with differential rates of profit.

Notes

* This paper was written while the author was staying at the New School for Social Research in New York as a visiting research scholar, and a shorter version was presented at the URPE (Union for Radical Political Economics) session at ASSA (Allied Social Science Associations) in Washington D. C., U. S. A. (December 30, 1990) Thanks are due to the valuable comments by Professors A. K. Dutt and H. D. Kurz at the Conference. The author was also much indebted to Prof. Willi Semmler for providing the comfortable research environment. Needless to say, however, any remaining error is the author's own.

(1) From eq. (3) the (pre tax) 'maximum rate of profit' in the case of zero real wage is obtained as $R=\{1-(a\Theta+\pi m\Theta)\}/(a+\pi m)$. If the system is productive, R must be positive so that the inequality $1>a\Theta+\pi m\Theta$ must be satisfied. From now on, we assume that this condition is in fact satisfied.

(2) Obviously, the terms of trade depend on the rate of foreign exchange as well as tne price level of the foreign country in terms of foreign currency. ***Other things being equal***, the rise (the fall) of the value of the domestic currency relative to the foreign currency will improve (deteriorate) the terms of trade.

(3) This procedure of the introduction of the tax into the present analytical framework is in line with Eatwell (1980).

(4) Fig. 2 is but a reproduction of Fig. F2 in Asada (1989).

(5) $B\geqq 0$ implies that the matrix (or the vector) B is nonnegative. $B\geq 0$ implies that B is semipositive, while $B>0$ implies that B is strictly positive.

(6) This assumption implies that all domestic goods are 'basics' in the sense

of Sraffa (1960). We can extend easily, however, the most of the results of the analyses in this paper to the case where the 'non basics' exist. See, for example, Pasinetti (1977) chap. 5.

(7) Contrary to the usual formulation of the Sraffian system, we do *not* assume the equal rate of profit among industries. In other words, we allow for the existence of some 'monopolistic' elements in the economy. As for the analyses of the differential profit rates among industries in the somewhat different context, see Semmler (1984), Steedman (1984) and Asada (1988).

(8) we denote the transpose of the matrix (or the vector) B by B'.

(9) The idea of this formulation is essentially owning to Ara (1987) chap. 12. See also Metcalfe and Steedman (1979b) and Steedman (1979).

(10) The Perron-Frobenius theorem assures that p is strictly positive and unique up to scalar multiplication since G is nonnegative and indecomposable *by assumption*. (As for the Perron-Frobenius theorem, see, for example, Nikaido (1968) chap. 2.)

(11) Steedman (1984) provides the similar argument in the context of the closed economy.

(12) *Assumption 3* assures that there exists $j \in \{1, 2, \cdots\cdots, s\}$ such that $m_{kj} > 0$ if $k \in \{1, 2, \cdots\cdots, n\}$ is fixed arbitrarily.

(13) It must be noted that the statements of *Theorem 1* are reinforced rather than invalidated if some imported goods are used both as the capital goods and the wage goods.

(14) $[I - C\ominus]^{-1}C = [I + C\ominus + \{C\ominus\}^2 + \cdots\cdots]\ C \geqq C \geqq A \geqq A\ominus$ and $A\ominus$ is indecomposable from *Assumption 2*.

(15) Note that we can also express eq. (31) as

$$\omega^* \equiv w\ell x^* / (p[I - C\ominus]x^*)$$

from eq. (30), so that ω^* can be considered to be the wage share in the (hypothetical) standard system.

(16) We do not pretend to assert that such a formulation is new or original. Similar models were already investigated by several authors. (See, for example, Okishio (1977b) chap. 1, Aoki (1977), Nikaido and Kobayashi (1978), Goodwin (1983) chap. 4 and Ara (1987) chap. 12.) We take up this model only as an important example of the application of the Sraffian idea to the simple dynamic system.

(17) Obviously, we are implicitly assuming that $\tilde{r}_j > 0$ for all j and $\tilde{\omega} > 0$.

(18) See Nikaido (1968) chap. II.

(19) This condition includes the pure circulating capital model ($\delta_1 = \delta_2 = \cdots\cdots = \delta_n = 1$) and the ever-lasting fixed capital model ($\delta_1 = \delta_2 = \cdots\cdots = \delta_n = 0$) as two famous extreme cases.

(20) As for such a procedure, see, for example, Nikaido (1968) chap. 2, Murata (1977) chap. 3 or Ara (1987) chap. 1.

(21) By assumption, G_1 is indecomposable. In this case, K (λ) is also indecomposable if $\lambda > 0$.

(22) It is easy to prove that H is indecomposable if G_1 is indecomposable. However, G_1 is not indecomposable if there are some domestic goods which are not used as the wage goods nor exported in exchange for the imported wage goods.

(23) For simplicity's sake, we ignore the international trade throughout this section.

(24) V is indecomposable because $V \geqq A\Theta$ and we are assuming that $A\Theta$ is indecomposable. Therefore, $[I - V]^{-1}$ becomes to be strictly positive.

(25) We can also trace the effects of the other kind of the fiscal policy or the monetary policy in a similar way. By the way, it must be noted that we might underestimate the instability of the system because we are neglecting the effect of the inflation expectation. If the inflation expectation is explicitly considered, the system will become more unstable.

(26) It is well known that even the simplest type of the nonlinear difference equation with single variable can produce the chaotic behavior. See, for example, Day (1982) (1983) and Bhaduri and Harris (1986).

(27) Nikaido and Kobayashi (1978) managed to analyze the interrelated price-quantity dynamics in a rather simple way, but at the cost of the unrealistic assumptions that (i) the profit is automatically invested and (ii) the proportions of the wage goods coincide with those of the standard commodity ***by accident***. In the models presented in this paper, we required ***no*** such assumptions. By the way, as for the recent studies of the so called 'cross dual' dynamics of some sort of the interaction between the price system and the quantity system, see Flaschel and Semmler (1987) (1988).

(28) Essentially the same formulation was developed by Okishio(1977a)chap. 2. This is an answer to the following question by Steedman. "How could

the traditional Marxist embodied labour content of commodities be determined in an open economy, when there is no way of allocating to individual commodities, produced with imported means of production, the labour used to produce the exports which 'pay' for those imports?" (Steedman (1977) p. 200)

(29) Note that the labor values in this model depend not only on the technological conditions but also on the terms of trade. For example, the improvement (deterioration) of the terms of trade induces the decrease (increase) of the labor values.

(30) Let us consider the dual problem of (A5), i. e.,

Maximize $v\omega(b^d+f\bar{\pi}b^f)N$ subject to $v[I-(A\Theta+f\pi M\Theta)]\leq \ell$, $v\geqq 0$. (*)

For any $v\in E\equiv\{v\in R^n_+ \mid v[I-(A\Theta+f\pi M\Theta)]\leq \ell\}$, we have $v\leqq \ell[I-(A\Theta+f\pi M\Theta)]^{-1}=\Lambda\in E$ since $[I-(A\Theta+f\pi M\Theta)]^{-1}\geqq 0$ by assumption. Therefore, Λ is an optimal solution of the problem (*) so that we have $\omega\Lambda(b^d+f\bar{\pi}b^f)N=\ell x^0$ in view of the duality theorem of linear programming.

(31) In this appendix, we are assuming, following the Marxian tradition, that wages are paid out of capital rather than out of the current revenue.

(32) From the equations (A1) and (A7) we have $p/w=\Lambda$ if $r=0$, where $w=p\omega(b^d+f\bar{\pi}b^f)$ is the money wage rate. This equality means that the 'commanded labor' is equal to the labor value if there is no profit. This fact means that the definition of the labor values in the forms of eq. (A1) passes a sort of the 'consistency test'.

(33) We assume that the matrix $\boldsymbol{Q}$ is nonnegative although r need not be nonnegative.

(34) T is nonnegative since $1+e>0$ from eq. (A3).

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